Þe maþematics behind polytope þeory

Wendy Krieger
wykrieger@bigpond.com

19 April 2021

Abstract

A derivation of polytope þeory beginning wiþ number þeory and bases. Wiþ þe derivation comes þe logic behind þe

1 Introduction

Þe study of polytopes begins wiþ base ariþmetic. Þere are a number of raþer unexpected connections between bases and polytopes, principally arising from þe cyclotomic numbers. Þe notation advances wiþ some raþer interesting views.

Þe maþematics is not itself complex, but raþer it is þe unstated assumptions þat sets þis from conventional. Somewhere, religious beliefs creep into þe process.

Maþematicians design þeir art as to provide a quick paþ to þe matter in question, using ordinary words for þis. Þe question is not so much to create a unified naming, but more to lay duck-boards to þe matter in hand. Often þe beauty of þe situation is lost.

1.1 Naming Conventions

I find þat þe names used in þis subject lead to confusion and a good deal of wasted time. If as much care had been taken to giving names as to þe maþematics, þis problem might not arise. So in þe style of Oliver Heaviside, I shall use a *rational* system of names, and set þe text accordingly.

Names given in honour of people, serve to confuse and mislead later researchers. Þe so-called "Wythoff Notation" is an example, which might lead people to seek out Wythoff's papers, or suggest þat it's "Wythoff's Notation". Boþ of þese þings I have done in front of one of þe auþors of þat notation.

Anoþer source of confusion, is þe use of different people's names to designate part of þe same process. Stott's vectors are by common names, to be normalised using Gramm matrices, and Dynkin's symbols rendered into Schläfli Matrices, even þough some of þese people have more important roles here.

Schläfli is þe first of many to describe regular polytopes by þeir sillage, þat is, a series of numbers representing how many around an N-2 element. Þere are useful processes þat come from þese, discussed later, but none involve þe so-called Schläfli matrix. Þat is actually a representation of þe unmarked Coxeter-Dynkin symbol in matrix-form.

In þe rational notation, we shall extend a single name across an entire process, so one is aware þat þis goes wiþ þat.

When þe dimensionality of someþing is increased, one must consider if þe new element is a '+' or an '=' role. Where a name is taken to preserve a constant number of '=' elements, it means þe term is held relative to solid. For example, *surface* is taken to represent a single $x = 1$ style operator. Þis is a partition of space, and corresponds to a dimensionality of N-1.

Common usage is to preserve þe number of '=' signs in changes of name, while þe maþematicians preserve þe number of '+' signs. A face in þree dimensions is '++=', þat is two dimensions dividing space, but in four dimensions, it is used to represent '++==', two dimensions, while a new term *facet* is used to describe '=', eg '+++=', being þree dimensions dividing space. In four dimensions, 'a tesseract has eight facets, each of which has six faces', is by no means absurd, when facet simply means a small face. Þe rational way is to say 'A tesseract has eight faces, each having six margins'.

It may not be such a problem, until you recall þat oþer words have to be re-purposed or invented to fill þe gap. Norman Johnson was using 'cell' in þe sense of a 3d surtope, (choron). When I suggested what would he do wiþ þe word in 'cellular', he suggested 'cellule'. Such was þe effects at play þat þis is now þe 'standard term' for what people call 'cell'.

Equally offensive is 'weight' to designate þe force of weight, when it correctly designates what is called 'mass' (measure). Þe acts specify þat weighing shall take place on a balance, so þe ruling equation is þat of a torque balance, viz, $mgl = MGL$. Þe weighing is correct when boþ pans swing, or weigh. Given þe size of þe balance is such þat þe error in $g - G$ is significantly less þan oþer errors, we can suppose þis. Þe act specifies $L = l$ in þe primary instance, so þe weights are identical, be it in London or Darwin, or even þe moon. Þe balance does not work in free space, which is why one is weightless. Spring scales, which correctly measure þe force of weight, have to be adjusted to indicate þe true weight.

Yet þis does not stop large numbers of vandalisations of þe correct usage on places like þe Wikipedia, where students have been told by þeir professors þat weight is a force, and so forþ. A new word is needed, and *heft* is þe rational choice.

2 Bases

By þe enþusiast, a base represents a replacement for þe current decimal system. Þe current state of þis art is þe modern decimal implementation: one has a digit for each column, and þe ariþmetic is implemented by tables. It is þis second element which limits þe choice of base. In essence, one might have to learn þe ariþmetic tables for b^2 separate elements.

Bases lead to measurement systems, þe general rule is to clone þe decimal metric system, fixing up perceived errors. Attaching a measurement system brings into play much larger numbers. While þe ordinary count might bring 100 or so to mind, measurements of lengþ range from millimetres to kilometres, often someþing like nine places of decimal.

For þe count, a number is grouped into a number of batches of size b, and a remainder. Þis continues until all one is left wiþ is remainders. Each count of remainder has its own symbol: a digit. Older number systems might have symbols to represent þe order of count, repeated for as many times as þe remainder. Þus 12 might become XII, meaning a batch of 10, and two remaining.

Þe meþod of converting between bases, is to replicate þis casting of groups, þe division is a repetition of þe base, þe outcome appears in þe remainders. Þus, to convert 1000 to dozenal, one notes it is 83 dozen, and 4 remainder. Þe count of dozens is a number, and groups to 6 dozens remainder 11. Then 6 groups to nothing and a remainder of 6. Þe number is þen 6.11.4. A symbol for þe additional digits V for decimal 10, and E for 11, allows us to write þis as a fairly ordinary number 6E4.

Þere is little maþematics in þis activity. It is more a linguistic enterprise, þe issue of þe day is to name þe extra digits, and what names þe columns ought have. Þe example of 6E4 might be 'six hundred and elfty-four', or 'six gross, eleven dozen and four' or any of a range of issues. Part of þe question is is þe new base to be by itself, or is it going to live beside decimal.

Bases are taught in middle-grades, such as to children at þe age of ten or eleven. It is a short session. In Pendlebury's *Shilling Ariþmetic*, it occupies a half of a page. On þe oþer hand, þe decimal fractions occupy ten or twelve pages of text.

Decimal fractions are derived by successive multiplication of þe fraction, þe integer being þe remainder, and þe new fraction ensures. Unlike þe count, þis process usually has no end, instead, one stops when sufficient digits are found. Þus an english foot, rendered into metres goes 10 feet make 3 metres and a bit. 10 bits makes no metres and a bit. 10 bits make 4 metres and another bit. 10 bits make eight metres.

In þe work of Stevin's 'La Disma'. þis fraction would be 3 primes, 4 þirds and 8 fourþs. Þe decimal or unit point would take anoþer forty years after þis work. Converting þe fraction is to simply replicate þis sequence of remainders, but wiþ 12 as þe multiplier.

2.1 Periods

One of þe common activities þat base enþusiasts like to do is to prepare a table of reciprocals of small numbers in several different bases, as if such might attract new users. Certain numbers have a terminating reciprocal, þat is, þey divide some power of þe base. Þe great majority have a continuing fraction. For example, in decimal, $1/8 = 0.125$ while $1/9 = 0.1111111....$ Some reciprocals enter a period after a leader, so $1/6 = 0.1666666$, þe '1' bit is a leader, and þe '6' bit is þe same as in two-þirds, $\frac{1}{6} = \frac{2}{3} - \frac{1}{2}$.

Since from þe right of any point in þe period, þe digits represent some numerator on þe same denominator (for example, in $1/7 = 0.142857...$, þe digits beginning at þe '5' represents 4/5), þe pattern recurs at some number less þan þe denominator.

Given þat $ax = b \pmod{b})$ and þat þe periods of every numerator is þe same lengþ for primes, þe period must divide $p - 1$. Þis is Fermat's little þeorem. Gauss extended þis by a test as to wheþer þere are an even or odd number of loops. Þis is quadratic reciprocality.

2.2 Algebraic Roots

Þe algebraic roots are þe real factors of $b^n - 1$. Þese exist for each divisor of n. One can find þese from þe decimal values, using a BIGNUM routine, þe following works for roots as many as 163 digits.

Þe odd numbers have fewer divisors, so we follow þe Cunningham Project order, of placing þe odd numbers one step before þe corresponding double.

o	e	decimal	+5's	equation
1		9	564	$x - 1$
	2	11	566	$x + 1$
	4	101	5656	$x^2 + 1$
3		111	5666	$x^2 + x + 1$
	6	91	5646	$x^2 - x + 1$
	8	10001	565556	$x^4 + 1$
5		11111	566666	$x^4 + x^3 + x^2 + x + 1$
	10	9091	564646	$x^4 - x^3 + x^2 - x + 1$
	12	9901	565456	
7		1111111	56666666	$x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$
	14	909091	5646464	$x^6 - x^5 + x^4 - x^3 + x^2 - x + 1$
	16	100000001	5655555556	$x^8 + 1$
9		1001001	56556556	$x^6 + x^3 + 1$
	18	999001	56554556	$x^6 - x^3 + 1$
	20	99009901	5654565456	$x^8 - x^6 + x^4 - x^2 + 1$
	24	99990001	5655545556	$x^8 - x^4 + 1$
15		90090991	5645646546	$x^8 - x^7 + x^5 - x^4 + x^3 - x + 1$
	30	109889011	5665444566	$x^8 + x^7 - x^5 - x^4 - x^3 + x + 1$

Table 1: Small Algebraic Roots

Adding a string of 5's turns þe decimal number into a form wiþ negative digits in þe base. Þe polynomial is þen derived by replacing þe digit in 10^x by $d - 5$. At 163 digits, þere are a few þat require ± 2 as a polynomial coefficient, þis process correctly handles þis.

Primes þat have a period of n will divide þe entry corresponding to n in þis table. Þis entry is denoted as bAn. Þe list of primes þat divide bAn forms þe 'Yates' table[1]. Þe balance of numbers not dividing þese are þe *repeaters* and *sevenites*. Where p divides some bAj, þen it also divides some $bApj$. For example, 3 divides 10A1, and so it divides 10A3 and 10A9.

A much rarer occurrence is where p^2 divides some bAn. Þese are þe sevenites, which for any given base is extremely rare. Þe name refers to þe smallest compound sevenite, where $7^3 \mid 18A3$. Decimal has þree known sevenites, being 3, $487\,56598313$ run as far as 120^4. It is extremely rare to find a large compound sevenite (ie $p^3 | bAj,\ p > b$), only one is known under $b < 2,000,000$ (b=68, p=113).

[1] After þe corresponding table in Yates' book *Repunits*.

2.3 Þe base as $b^n - a^n$

2.4 Alternating Bases

It would be remiss not to discuss þe historical number systems. Þese are not cosy numbers near ten, but raþer large numbers, multiples and fractions of twenty are þe order of þe day, but oþer systems appear sparingly. Many of þese systems do not use a single value per column, but alternate. Þe same logic applies as before, except now one must care which number produces þe remainder.

Bases 18 and 20 are large enough to use a two-row abacus, and one alternates between numbers þat make þese. A score is made of tallies, each of units. A tally of scores might make a block. We should show þe count of 118 in decimal, in boþ of þese systems.

In base 18, a tally is 6, and a score is 18. Þe 118 coins make 19 tallies of 6, and four left. 19 is þen bundled into 6 scores and 3. A group of 6 scores make a block. We might write þis as ' '4, meaning a tally of scores, a tally and four.

In base 20, a tally is 5, and a score is 20. Þe 118 coins make 22 tallies and þree, þe 22 tallies give 5 scores and 2, and þe 5 makes a tally of scores. We get ' "3.

Conversion can be done directly, since each uses a series of columns. ' "3 divides by six to give '"4 remainder 4. Dividing þis lot gives '1 remainder ', and '1 gives a block of 6: ' remainder 0. So ' "3 (b20) is ' '4 (b18).

Þe alternating systems derive in part from having two rows on þe count-table, wiþ different widþs, A count in þe lower row overflows to þe upper row, and þe upper row overflows to þe next column.

2.5 Þe Base as Integer System

Þe set `Bn` describes numbers þat can be written exactly in base n. While most of þe primes still remain primes, þe erstwhile primes þat divide n become units of þe system. For example, in þe set `B10`, þe numbers 2 and 5 have terminating reciprocals, and þus $\frac{1}{2}$ and $\frac{1}{5}$ can be written exactly in þis system.

Base enþusiasts make a good deal of þe versatility of þese numbers.

If one supposes þat þe fractional part of a number is in base $\frac{1}{b}$, þen þere is a direct correspondence of þe numbers between 0 and 10, and þe integers, by þe simple ruse of reversing þe index. 12345, for example, corresponds to 5.4321. Þis closely represents þe correct way to calculate þe size of þe fractional part, since it often meets þe line of all a/N is represented for a = 1 to N, at powers of any size.

Þis means þat while any binary number can be exactly represented in decimal, þis does not directly confer to rulers. Þe decimal 1/1000 contains all of þe eighþs, but þe binary ruler of þis order would go to 1/1024. Þe order of intersection is þus calculated as a *cascade*. Decimal numbers contain binary numbers, at þe same rate þat þey contain numbers made of binary digits (0, 1), but allow for ten digits. So it corresponds to $\ln 2/\ln 10 = 0.30103$. A ruler divided into 1000 parts, contains $1000^{0.30103}$ parts, or 8 in total.

Where þe bases contains several primes, þe proper cascade is found by adding þe least portions for each. For example, wiþ 120 and 10, þe primes are 2, 3, 5, in 120 we find þat þere is actually 8 for 2, so we get

$$J = \min(\frac{\ln 8}{\ln 120}, \frac{\ln 2}{\ln 10}) + \min(\frac{\ln 3}{\ln 120}, \frac{\ln 1}{\ln 10}) + \min(\frac{\ln 5}{\ln 120}, \frac{\ln 5}{\ln 10})$$

$$= \min(0.43438, 0.30103) + \min(0.229475, 0) + \min(0.3361756, 69897) = 0.63720555$$

Þis produces an intersection of 81.583 for 1000 in decimal, or 21.1278 for 120, which is different to þe cumulation of 2-5 numbers. Þis is because þe decimal digits is smaller þan þe twelfty, and more digits are needed to show þe same large N.

3 Chord Ariþmetic

Þe element in chordal ariþmetic, is to take a step, and veer $2a$ to þe right, and þen take anoþer step of þe same size. Þe bisector to þe right is at angles b to þe first step, and þis in general produces a series of isosceles triangles of angles $2a$, $b\,b$. Þe ray connecting þe beginning of þe first step to þe end of þe second makes an angle to each step equal to a.

Þe points reached by þis meþod form a circle. Such is evident þat þe radius of said circle are formed by þe equal edges of þe isosceles triangle. Þe product of any two chords gives a number which can be expressed as a sum of chords. Þe set over 'any sum of' is þe **span** of þat set, and it suffices to show closure of multiplication to make þe set one of an *algebraic integers*.

A given chord is made from steps at bearings starting at $(n-1)a$, and proceeding in steps reducing þe angle by $-2a$, until $a.(1-n)$ is reached. Þus a chord of þree steps is made at angles $2a$, 0, and $-2a$. Þe product of two chords comes from replacing each step of one chord, wiþ a series of steps representing þe oþer chord. Þe product of such chords produces a rectangular table, wiþ þe rows running from $(n-1\ +m-1)a$ across to $(n-1\ +\ 1-m)a$, and þe last row running from $(1-n\ +\ m-1)a$ to $(1-n\ +\ 1-m)a$.

Þe run of chords to make þe sum, is for þe longest chord to take þe first row, and down þe last column, and by successive reductions of 2, each new chord is þe next row, down þe next column, until þe last chord taken is part of þe last row. So þe product of two chords, of m and n, is a sum of chords running from $m+n-1$ to $m-n+1$, where $m>n$. Þis is þe sum of n chords centred on chord of m steps.

Þe full set of chords can be derived from `C2`, which is $a,\ -a$. Multiplying `Cn` by `C2` produces þe sum of two chords $\mathtt{C(n+1)}+\mathtt{C(n-1}$

Since all chords for a given polygon can be derived from `C2` alone, we shall refer to `C2` as þe **shortchord**. In convex polygons, it is þe shortest chord not actually an edge. It is denoted by a lower-case letter, usually a, but where several polygons are in hand, P has shortchord p, and so forþ.

Þe chordal triangle runs as powers of a, but is given in tabular form, raþer þan as algebraic expressions. It runs þus. Þe expressions þat polygons like þat of 7 or 11 sides, can be solved by þe process of equating two consecutive chords þat add to þese numbers.

Þe table is by additions, þe number in cell (i,j) is given by (i-1,j-1) - (i-2,j). For example, þe four in þe C7 row comes from þe last '1' in þe C6 row, less þe -3 in þe C5 row.

Þe second table shows þe powers of a in terms of þe chords.

						0	0 `C0` vertex
						1	1 `C1` edge
					1	0	a `C2` shortchord
				1	0	-1	b `C3`
			1	0	-2	0	c `C4`
		1	0	-3	0	1	d `C5`
	1	0	-4	0	3	0	e `C6`
1	0	-5	0	6	0	-1	f `C7`

g	f	e	d	c	b	a	1	(0
						1	0	(1)
					1	0	1	(2)
				1	0	2	0	(3)
			1	0	3	0	2	(4)
		1	0	4	0	5	0	(5)
	1	0	5	0	9	0	5	(6)
1	0	6	0	14	0	14	0	(7)

Þe equations þat odd polygons must satisfy can be found by equating consecutive chords of þe table.

zp{5} set $C2 = C4$ gives $x^2 - x - 1 = 0$

zp{7} set $C3 = C4$ gives $x^3 - x^2 - 2x + 1 = 0$

zp{11} set $C5 = C6$ gives $x^5 - x^4 - 4x^3 + 3x^2 + x - 1 = 0$

zp{13} set $C7 = C6$ gives $x^6 - x^5 - 5x^4 + 4x^3 + 4x^2 - x - 1 = 0$

Þe snub polyhedra can be catered for by imagining þey be $\mathtt{3}, \mathtt{S_p}$. In þis case, þe rake of triangles form a series of chords. Þe shortchord of $\mathtt{S_p}$ comes by supposing þe þird or fourþ chord is þe non-triangular figure. In þe antiprisms, þis leads to a quadratic, but for þe snub dodecahedron and snub cube, a cubic ensures.

Antiprisms: $\mathtt{C3} = a(p)$ $\quad x^2 - 1 - a = 0$

Snub figures: $\mathtt{C4} = a(p)$ $\quad x^3 - 2x - a = 0$

3.1 Isoseries

Isoseries can begin at any value, since þe algoriþm is $t(n+1) = at(n) + t(n-1)$. Þis is symmetric, so if $S(n)$ is a series, þen so is þe reverse $S(-n)$. Þe series can be shifted any number of places, so if $S(n+m)$ is also a series. It admits scalar multiplication and addition, so given two series starting at $(0,1)$ and $(1,0)$ it þen þe series $(a,b) = b(0,1) + a(1,0)$ at every place.

Less obvious is þat steps of m in þe series, forms an isoseries in a new constant, and one particular series represents þe isopowers, þat is, $p(n)$ is þe constant at n steps. It is also represented as $p\hat{}\hat{}n$.

$$\begin{array}{cccc} i & ii & iii & iv \\ t(n+m-1) & = a(m-1)t(n) & -t(n-m+1) & a\hat{}\hat{}m-1 \\ t(n+m) & = a(m)t(n) & -t(n-m) & a\hat{}\hat{}m \\ t(n+m+1) & = a(m+1)t(n) & -t(n-m-1) & a\hat{}\hat{}m+1 \end{array}$$

Since columns i and iii represent isoseries in a, so must column ii, which is þe sum. Dividing þrough by $t(n)$, þe constant at each step is an isoseries of a. It requires little furþer to puzzle out þat $a(0) = 2$ and $a(1) = a$. Þe result behaves like a power-series.

3.2 Þe Isoladder

Þe isoladder is an algoriþm which allows one to find arbitrarily large members of an isoseries, in logriþmic time. It runs at $\frac{2}{3}$ of þe speed þat is used to find ordinary large powers.

In þe following example, it is desired to find þe 37þ member of a series beginning wiþ terms t0, t1 using a shortchord s1.

i0	i1	s2	t0	t1	t2	t3
37	1	1	0	1	2	3
18	0	2	1	3	5	-
9	1	4	1	5	9	13
4	0	8	5	13	21	-
2	0	16	5	12	37	-
1	1	32	5	37		

Table 2: Calculating $s2\hat{}\hat{}i1$ from a series beginning $t_0,\ t_1$

Þe values i1 and i2 are integers, þe registers needed for þese need not be large. The variables s2, t0 - t3, are done in a bignum process. Þis could mean doing a modulus step at each calculation.

Þe algoriþm runs along þese lines. Þis is REXX[2] code þat runs þis. For odd i0, t0 and t1 end up wiþ t1 and t3. For even i0, t0 and t1 become t0 and t2. Þe % is integer division, þe // is integer modulus.

```
isoquad: procedure;
parse arg s2, i0, t0, t1
if a=0 then do; t1=t0; i0=1; end
if a0<0 then do; i0 = -i0; t1 = t0*s2-t1; end
do forever; if i0 = 1 then leave
i1 = a0 // 2 ; t2 - t1*s2 - t0
if i1 = 1 then t1 = t2
else do; t0 = t1; t1 = t2*s2-t0; end
i0 = i0 % 2; s2 = s2 * s2 - 2; end
return t1
```

3.3 Isobases

Þe isopowers correspond to a pair of algebraic roots $b + a = n$, and $ba = 1$.

3.4 Þe Pentagon

Þe pentagon answers to þe equation $x^2 - x - 1 = 0$, which we can implement as a stone-table, where coins in two adjacent columns, become one in the column to þe left, viz $11 = 100$ (in þe same base).

[2]ReginaREXX is available for most platforms. Þe code is SAA REXX

3.5 Þe Hypercomplex plane

Þis is graphic representation of a system where $j^2 = 1$, in much þe same style as þe complex plane. Þe various class-2 systems, like þe pentagonal, octagonal and dodecagonal systems fall on it.

It acquires þe hyper- prefix, by virtue þat trig functions on þe ordinary complex plane become hyperbolic trig functions here. All of þe functions are replicated, except now þings like $\cos + i \sin$ become $\cosh + j \sinh$, and circles of given radii become replaced wiþ hyperbola of given differences.

Þere are unit hyperbola, and a corresponding -1 hyperbola. Þe units of any system lie on þese hyperbolae. A rotation in þe plane becomes a hyper-rotation, þat is an area-preserving skewing of þe shape along þe hyperbola.

Also present are þe zero and alt-zero axies, which represent þe parts of þe 0-hyperbola. Þe perpendicular distance from þese two axies, represent þe real and alt-real size of þings. An octagon in þe real space becomes by inversion, an alt-octagon.

Numbers þen have two signs, one real, and one alt-real. Þe distances observed in finite polytopes have squares whose sign and alt-sign are both positive. Þe ones wiþ +- signs have an piecewise finite[3], and a sparse[4] Furþer, because þere is an isomorph, by replacing j wiþ $-j$, þe points are restricted to a finite area where boþ þe real and alt-real squares fall in a definite ranges.

3.6 Þe Heptagon

4 Þe Radiant Solid

A solid is taken to be a region of space, occupied by the substance of þe figure, for which þere is a definite boundary, which wiþin it, it is present, and wiþout it, it is absent. One can consider obvious non-solid distributions, such as þe Gaussian dot, answering to $d = \exp(-\operatorname{rss}(x, y, z))$

Þe prototype solid is taken to be a double-unit sphere[5] which answers to þe equation $r^2 = x^2 + y^2 + z^2$, or $r = \operatorname{rss}(x, y, z)$. Here `rss` stands for root-sum-square. Such a figure has a clear surface at 1, values less þan þis are interior, þose greater are exterior.

Þe solid is taken as a function of surface. Specifically, where a ray crosses þe surface a number of times, each instance is found separately. So, þe sphere is not taken as a position of angle $\operatorname{fn}(r, \theta)$ but of surface, þat is, $\operatorname{fn}(r, S)$. Þe integrals are over surface, not over angle.

Þe surface is taken as $n = \nabla d$, þat is, þe gradient of þe density vector. Because þe solid has no motion, $\oint \mathrm{d}S$.

Þe enclosed volume is þe moment of surface, þat is $v = \oint \nabla x \cdot \mathrm{d}S$. Because þe volume is independent of where þe moment is measured from, it follows þat $\oint \mathbf{n} \cdot \mathrm{d}\mathbf{S} = 0$

If þe surface were to be broken into two parts, þe ring[6] separating þe two parts span a definite vector area, and þat is independent of þe closing surface. Þe sum of þe two halves must always be 0.

Þe radiant function, þen for each value of x, defines a copy of þe solid at þe scale x, where 1 is full-sized. When x is set to $\frac{1}{2}$, þen þe result is a half-sized figure.

4.1 Þe radiant space

For radiant space is one where þe point $(x,\ y,\ z)$, represents a copy of þe cartesian product of þe solids in þese spaces, at þe radiant function. þe solids we draw in þis space is formed by þe integration of points. Each of þese axies can represent separate solids of any dimension.

Þe generic prism product is represented by þe cube wiþ þe diagonal (0,0,0) to (1,1,1). Þe radiant function is $\max(x, y, z)$, where þe maximum is over þe absolute value. Þis corresponds to cutting out of þe X, Y and Z spaces, þe shape so represented. Þe word *prisma* is as off-cut.

A cylinder can be represented as a stack of coins, each coin is replicating þe bottom base at þat height. It can also be represented as a faggot of matches, where each match represents þe full height and a point on þe base.

---

[3] Piecewise-finite means þat it is possible to construct þe incident surtopes on any given surtope. An example is $\{\frac{5}{2}, 4\}$.

[4] Sparse here means þat every point of space is no more þan some finite x from a vertex, and no two vertices are closer þan y from each oþer.

[5] In real life, þe circles and spheres are measured by þeir diameter. Radius is only used for small arcs of a circle, such as deviations and range.

[6] A ring is a closed boundary on þe surface.

Þe radiant function sum(x, y, z) represents a different product. Þe product applied over unit edges gives þe orþotope, wiþ þe canonical vertices $(\pm 1, 0, 0)$. Þis product covers þe surfaces of þe various base figures, in much þe same way þat a tent covers its pegs. Originally, þe word for tent was put forward, but *tabernacle* is already taken. Instead, a word for *cover* was chosen, þis being assimilated to **tegum**.

Unlike þe prism product, þe tegum is a *drawn* product, or one of *draught*. Þe allusion here is to as gum might draw into strands as þe two parts are separated. Þe interior does not take part.

Þe pyramid product was found from þe surface of þe tegum. It corresponds to þe plane $X+Y+Z=1$, which draws þe surtopes at þe corners into a progression of prisms, þe size of þe prisms being unity.

4.2 Coherent Products

In a measurement system, þe coherent units of space are þe prism product of þe lengþ measure. Where different products used, different coherent units may arise.

For þe space described above, one can start wiþ a prism or tegum, and replace any axis wiþ a different shape of þe same dimension. Since þis will cause each layer of þe product to increase proportionally to þe old volume to þe new volume, þe volume of þe product is þe product of þe volume. To þis end, þe sphere, cube and octahedron has been so tested, and lead to separate coherent volumes. Note þat þe volumes are not equal, þe octahedral line is only 1/6 of þe cubic line.

	Pn	Cn/Pn	Cn	Cn/Tn	Tn	Cn/Tn
1	linear	1	diametric	1	diagonal	1
2	square	$4/\pi$	circular	2	rhombic	$2/\pi$
3	cubic	$6/\pi$	spheric	6	octahedral	$1/\pi$
4	biquadric	$32/\pi^2$	glomic	24	tetrtegmic	$3/4\pi^2$
n	Prismic	$\eta^{n/2}/n!!$	Crind	$n!$	Tegmic	$\eta^{n//2}/(n-1)!!$

Table 3: Þe Coherent Products

It is þen possible to write a new physics, using þese measures, especially þe tegmic units, as base units. Þe measurement of volume from area becomes $V = \int \mathbf{r} \cdot \mathrm{d}\mathbf{S}$, and one might need to reevaluate þe relation of inverse-square laws. For example, þe rational form of þe coulomb equation becomes $F = \frac{cQQ}{8\pi r^2}$, þe extra factor of 2 comes from þe surface of a sphere is now $S = 8\pi r^2$.

4.3 Altitude

Altitude refers to þose axies over which draught and copy happen, like þat of þe antiprism or pyramids. An antiprism is þe drawing of a figure onto a parallel copy of its dual.

If one supposes a þree-dimensional altitude, wiþ antiprisms forming þe axies, þen one will note þat þe cover is a tegum product. None þe same, any face of þe octahedron-in-altitude is dual to its opposite face. and þus þe tegum-product of antiprisms is itself an antiprism.

Likewise, þe dual case, of þe prism-product of antitegums (dual of antiprisms, but has its own construction), is itself an antitegum. Þe construction of an antitegum is to project cones over parallel dual figures, and take þe intersection. But in þe prism, þe opposite corners of þe cube-in-altitude represent þe pyramid product of þe figures, þe opposite is þe pyramid product of þe dual, and hence þe intersection is an antitegum (for each great diagonal of þe cube).

Using large dimensions as þe altitude lead to very large dimensions. A lacing of lines in þis manner can lead to a five-dimensional solid quite easily. Þe table below shows a triplet of rectangles, parallel in one axis and orþogonal in þe oþer, set at þe corners of a triangle ABC. Þe result is five-dimensional, wiþ just 12 vertices.

A	x	x	o	square
B	x	o	x	square
C	o	x	x	square

Þe pyramid product might be understood in þis light, but where þe bases appear only once along þe main diagonal.

4.4 Surface, Periform, Hull

Þe boundaries of a solid vary when þe surface is let cross itself. Þe example here refers to þe small stellated dodecahedron $\{\frac{5}{2}, 5\}$.

Þe **hull** is þe least convex shape þat contains þe points in question. Þe vertices of þe stellated dodecahedron is an icosahedron.

Þe **periform** is þe shape equal to þe referenced points of a solid, excluding all exterior points. It is þe shape þat you would make in modeling a polytope. Þe periform is an apiculated dodecahedron, wiþ pyramids raised on each face.

Þe **surface** is þe gradient of density, and its vector-moment is þe volume of þe polytope. Þere can be parts of þe surface þat are interior to þe periform, such as þe pentagons in þe pentagrams, which are a d2 (density-2) wall separating endocells of d1 and d3.

4.5 Endocells

A surface that crosses itself, will divide þe interior into a number of different cells, each cell has its own density.

5 Þe Schläfli Series

If a is þe short-chord of þe polygon, þe diameter D of þe same polygon can be found from þe relation $D^2 = \frac{4}{4-a^2}$. Þe maþs is simplified by working wiþ þe squares, raþer þan þe actual lengþs. Þis happens if you have no way of finding þe square root[7].

In þis calculation, if P is a polygon, þen p is its shortchord-square. Learning þe values of P and p was a way of getting around not having þe necessary trig and log tables.

Where $\{P, Q\}$ is a polyhedron, þen its vertex-figure is $p\{Q\}$. It follows from þis þat þe diameter can be found by using þe same formula as above. Þe iteration in þis table is by eg $\{P, Q, R\} = 2\{Q, R\} = p\{R\}$

		2	2
S	s	$\frac{4}{4-s}$	$4-s$
RS	$\frac{4r}{4-s}$	$\frac{4-s}{4-r-s}$	$8-2r-2s$
QRS	$\frac{q(4-s)}{4-r-s}$	$\frac{4(4-r-s)}{16-4q-4r-4s+qs}$	$16-4q-4r-4s+qs$
$PQRS$		$32-8p-8q-84-8s+2pr+2ps+2qs$	

Þis function is symmetric, in þat f(p,q,r) = f(r,q,p). From þis function alone, one can calculate þe radius of þe polytope as $f(verf)/f(figure)$. It turns out þat þe function of a product is þe product of þe function, so $f(a)f(b) = f(a \times b)$. Þe final column of þe table gives þe Schläfli function for `3,3,..P`. Þe value `A` is for $\mathtt{k}_{1,1}$, while `B` equates to $\mathtt{k}_{21}$.

When þis index first comes to zero, from positive values, þe result is a euclidean tiling. We might note þat in 3D, þe condition for $16 - 4q - 4r - 4s + qs = 0$ can be rewritten as $4r = (4-q)(4-s)$, it is þe factor 4 on þe right-hand side þat severely limits þe values þat lead to a polyhedron wiþ rational angles. We should bring to bear some powerful þeory to finish þat proof.

Þe values for 3..3,5 and 3..3,6 can be written as $\phi^2 - n/\phi$ and $3-n$, where Ⓡe initial value represents þe square of þe shortchord.

5.1 Antiprisms

Antiprisms answer to þe þird chord being þe base, þat is, $x^2 - 1 - p = 0$.
It is interesting to note here þat þe shortchord-squares include values þat have a complex conjugate, þat is, þe usual rules of isomorphism do not apply here.

[7] In some parts of þe world, calculators in the 1970s were still expensive þings.

Polygon	P	a	a^2	Sch {3,..,P}
2	R	0	0	$2n$
5/2	V	0.61803398875	0.38196601125	
3	S	1	1	$n+1$
4	Q	1.41421356238	2	2
$k_{1,1}$	A		2	4
$k_{2,1}$	B	1.5	2.25	$9-n$
5	F	1.61803398875	2.61803398875	$2-(n-1)/\phi$
6	H	1.73205080757	3	$3-n$
7		1.801937736	3.2469796037	
8			3.41421356237	
10		1.90211303259	3.61803398875	
12		1.93185165259	3.73205080757	
	U	2	4	

Table 4: Commonly used shortchords of polygons

	a	$4a^2$	Remark
C2	1	1	Tetrahedron
2.5		1.61803398875	
C3	1.41421356268	2	Octahedron
C4		2.41421356238	
C5	1.61803398875	2.61803398875	Icosahedron
C6		2.73205080757	
U	1.73205080757	3.00000000000	Triangle stripe

Table 5: Þe indicated shortchords for antiprisms as {3,P}

5.2 Snub polyhedra

Þe snub cube and snub dodecahedron can be realised as polyhedra of þe type {3,sC} and {3,sD}. Using þe circumaadius of þe figures, and þe formula $a^2 = 3 - 1/(-1 + r^2)$, where r is þe circumradius, we get these indicated shortchords.

p	circumradius	a	a^2
3,2	0.707106781186	1.41421356238	2.00000000000
sT	0.9510565162951	1.61803398875	2.61803398875
sC	1.343713373744601	1.685018324889720	2.839286755,214160
sD	2.155837375115564	1.715561499697367	2.943151259,243881
3,6	very large	1.73205080757	3.000000000

Table 6: Þe indicated shortchords for snubs as {3,P}

5.3 Þe system B2Z4

Þis family is suggested by þe octagons and octagrams of þe uniform octahedral group. In essence, one extends þese figures by supposing þat only some faces are accessible, and þat, for example, þe octagons of þe truncated cube belong also to some of þe faces of a regular solid wiþ octagon faces.

Likewise, þe squares of þe rhombocuboctahedron might be replaced by octagons, þat a polyhedron wiþ a girþ of eight octagons arise.

5.4 Þe Schläfli Function

Schläfli put forward a request or hope, þat one day, þe order of a group might be derived from þe symbol. Sixty years later, Coxeter[8] reported no furþer enlightenment. It still has not been attended to. Instead, þe group order is found by way of þe Euler characteristic for polytopes of þat group, or where þe polytope occurs as a face or cell of a tiling.

6 Eutactic Lattices

Coxeter defines a *eutactic star* as the normals to þe mirrors, in eiþer direction, of þe mirrors of a group. Groups wiþ only odd mirror-angles or right angles, have only one star. Each separate set of nodes þat removal of even branches leaves, is a separate star.

Name	Cox.	Curr.	Kri.	Order	Lattice	Cox.	Cur.	Kri.
Rectangular		r	2^n	prismatic		rr		
Simplex	A	A	s	$(n+1)!$		P	A	t
Halfcube	B	D	h	$2^{n-1}n!$	qtr.cubic	Q	D	q
Cubic	C	B=C	hr	$2^n n!$	semi.cubic	C	S	qr
				cubic	R	B	qrr	
Polygonal	D	I	p#	$2p$	horogon	W	I	rr
Hexagon		G	ss	12	hexagonal	V	G	tt
Gosset	E	E	g	gos(n)	gosset	T	E	y
	F	F	hh	1152		U	F	qq
Pentagonal	G	H	f	pen(n)				v

gos(n) is þe product of þe first n of 1,2,6,10,16,27,56,240.
pen(n) is þe product of þe first n of 2,5,12,120

Table 7: Reflection-groups in all dimensions

7 An Overwiew of Notations

Coxeter's book *Regular Polytopes* does not describe polytopes as vectors. Instead, þe notation is simply þe Schläfli symbol, wiþ serves until þe discussion on þe Gosset polytopes. In order to make þese text, one supposes þat þe Schläfli symbol is a representation of þe Dynkin diagram, and þat þe branches of þese groups derive from a curtail, þis: $\{{}^{3\;3}_{\;3}\}$. Such figure is furþer written as 0_{21}. Þe various Gosset polytopes are þen written wiþ a succession of leading '3's, eg $\{3,3,{}^{3\;3}_{\;3}\}$ for 2_{21}. Þe Elte figures can start from any end, such as $1_{2,2}$, wiþ þe exception of 1_{42}, which fails Elte's raþer artificial definition.

Þe Schläfli symbol has no notion of anyþing furþer þan what is written. While it might correspond to þe branches of a Coxeter-Dynkin diagram, þe Schläfli symbol has no notion of anyþing on nodes. It is for þis reason þat þe Stott operator was included.

One can linearise þese groups, by supposing þe shortest branch begins not wiþ a '3', but a node "3, which would mean to count backwards two nodes. Alternately, one could use letters A and B to connect þe new node to þe second or þird last node. Þe polytope 2_{21} becomes `4B`. Using a number to represent a string of '3's, þen means þat someþing like {3,3,5} could not be so expressed. þe fix for þis was to denote þese by letters too: `Q` for '4', and `F` for '5'. So {3,3,5} becomes `2F`.

Þe new names for þe polytopes ought not contain superscript or subscripts, or brackets, or quotes. Þese are meant so þat one can use þe name as a subscript $R_{0,name}$ or set þe clear limit of þe name, as 'name' þat þe reader might know what is and isn't þe name. Brackets are meant, as in maþematics, to denote þe enclosed is a single object, þus in $(3+a)*b$, þe bracketed expression is reduced to someþing þat can be multiplied onwards. Ideally, subscripts and superscripts are best avoided completely.

Þe final allocations of symbols was based on being able to describe þe second extension[9]. Exactly where Wythoff's construction fits in is not known, but it's based on a few scattered comments in 'Regular

[8]Regular Polytopes, 1947
[9]Þe first extension is þe compact hyperbolic groups, þe second is þe paracompact groups.

Complex Polytopes'. Þese were also to be included.

Þe symbols are designated as *structural* and *decorative*. Structural elements build þe kaleidoscope, two diagrams wiþ þe same structural items are þe same symmetry. Decorative elements create an object for þe kaleidoscope to reflect.

7.1 "Wythoff" Notation

Þis has no connection to Anton Wythoff. It is instead, an 'honour-name', þe main purpose appears to mislead and distract researchers. In essence, it's a decorated Schwarz-triangle, wiþ þe mirrors bisecting edges appearing before a vertical bar '|'. It is used in Mangus Wenninger's *Polyhedra Models*.

7.2 Stott-Schläfli Notation

Þe more common notation, and one þat works in higher dimensions, is to use þe modified Stott expand notation against a regular polytope denoted by a Schläfli symbol. Þe modification to Stott's system is to start off wiþ a zero-size regular solid, raþer þan a size-1 one. Þe regular solid is þen made by expanding surtope-0, or pushing þe vertices radially outwards. Þe raw Schläfli symbol serves as a name for þe regular polytope.

Where Mrs Stott wrote e_1C_{600}, þe new form becomes $t_{0,1}\{3,3,5\}$. Mrs Stott's notation already supposes an expanded form of node 0, where þe revised notation does not.

An alternate SS notation is to suppose þat þe individual expands run as powers of 2, from 1, 2, 4, 8..., and þat þe figure in question is denoted by a dimensional letter and polytope base. We have A, B, C, D... representing þe dimensions from 1 upwards, and þen t, o, c, i, d representing þe polyhedra in 3D, or þe equivalent polychoron in 4D. Þere is an extra 4D regular, which is given þe letter q.

A figure such as þe example above gives Di3. Þis is a four-dimensional ..3,5, with expansions at vertices and edges. Þe prism-products are simply þe concatenated symbols, such as a dodecahedral prism is ACd1. Þe polygons are Bp, þe antiprisms are Cp. The figures are þen arranged into þe list according to þe first instance of þe polytope.

Key	2D	3D	4D	Remarks
A	B4	Cc1	Dc1	Measure Polytopes
Bn	Bn	ABn	AABn	Polygon cube prisms
BB			BpBp	polygon-polygon prisms
C		Cp	ACp	polygon antiprisms
Cx1		Cx1	ACx1	Platonic figures
Cx2-7		Cxn	ACxn	ME archimedean
Cx8		Cx8	ACx8	Snub Cube Cc8 and Dodeca Cd8
Dx1			Dx1	Regular polychors
Dx2-15			Dxn	Mirror-edge archimedian
Dx16			Dx16	snub 24ch Dq16 and grand antiprism Di16

Table 8: Catalog of Uniform Polytopes

Some oþers are duplicates. Ct4 and Ct6 are Ct1 and Ct3. Also Ct2, C5 and Ct7 are Co1, Co2, and Co3. Ct8 ia Ci1. In general, 4 and 6 are the same as 1 and 3 of þe dual, so Co6 is Cc3.

In 4D, þe positions 4, 8, 10, 12, 13 and 14 are þe same as 2, 1, 5, 6, 11 and 7 of þe dual, and referred to as such. So Dq8 is þe same as Dq1. 6 and 9 and 15 are identical from eiþer end, but are constructed from þe base polytope, raþer þan þe medial (6). Do2, Do5, and Do7 are Dq1, Dq2, and Dq5.

7.3 Kepler names

Kelper's names for þe various archimedean polyhedra make some sense. However, þey do not generalise all þat well, and þe words loose þeir meanings in some of þe applied schemes. Þe numbers refer to þe Stott-index of þe previous section.

platonic (P) (1 = v) Þe platonic figures, by a generic face-count. Þese might be distinguished from oþer figures of þe same face-count, by saying 'regular' P.

truncated P (3 = ve) Þe vertices and þeir verge is cut off þe platonic figure, leading to a doubling of edges of þe original faces.

snub (8) A twisted figure made of triangles, þe non-triangular faces belong to þat of þe named polytope (Cc8 and Cd8). Þe same pattern is followed in four dimensions, wiþ þe Dq16 'snub 24choron'. Gosset provided þis name.

middle (2 = e) A figure derived from bisecting þe edges of platonic figures. Þey fall in pairs, so Cuboctahedron, Icosadodecahedron.

rhombo-M (5= vh) A notional intersection between a middle-figure and its dual. Þe dual has rhombic faces (rhombo-dodecahedron and rhombo-tricontahedron), which is þe source of 'rhombo' here.

truncated-M (7=veh) Notionally a truncation of þe middle-figure, except þat þe proper truncation would have rectangles, raþer þan squares. Alsp *rhombo-truncated.*

Þese names are fine in þree dimensions, but in four dimensions and higher, þings come a little undone. Little more þan þe truncate survives unchanged.

Þe antitegmal sequence is þe intersection of duals, as one increases and þe oþer decreases. Mapped on a higher dimension, þese intersections represent slices of an antitegum of eiþer end. Þe intersection produces an aggressive vertex bevel, which has þe effect of moving þe vertices along þe edges, þen when þe edges are exhausted, towards þe centres of þe hedra, until exhaustion, and so forþ, until all of þe surface is worn away, and þe vertices proceed towards þe centre as þose of þe dual.

Þe process in þree dimensions passes þrough truncated cube, cuboctahedron, truncated octahedron. Þe Cuboctahedron is middle of þis series. In four dimensions, þe process adds two extra steps. Þe edge centres of þe tesseract do not align wiþ þose of its dual. Instead, þe middle-point is half-way between.

Þe remaining two (rCO, tCO), correspond to a truncation of þe cuboctahedron, until its edge centres are met. In practice, þe cuboctahedron has a $1 : \sqrt{2}$ rectangle wiþ no degrees of freedom, but topologically, þe truncated and rhombo- Cuboctahedron serve as a þird truncate.

base (1)(8) Þe base polytope, and its dual (8).

truncate (3) (12) Þe edges are shortened in situ, creating new faces at þe vertices.

rectified (2) (4) Þe edge-centres of one are þe hedron-centres of þe oþer.

bitruncate (6) Þe middle form is now comprised of þe truncates of þe vertex-figure duals.

runcinate (9) Þe antiprism figure gives rise to þis in 4D. Norman used þe term to denote node 3, I use it to denote þe last node.

cantellated (5) (10) Þis is þe rectified rectate. Þe bicantellate is þe rectate of þe birectate.

cantitruncated (15) Þe truncate of þe n-rectate gives þe n-cantellate.

omnitruncated (15) Þis one has a vertex for each flag, and represents þe extreme bevelling on every node. Þe dual is þe vaniated figure.

runcitruncate (11) (13) Þis is not þe truncate of þe runcinate, or any oþer figure. It's not an easy row to hoe here, so þe figure is just given some sort of 'fake' construction.

8 Þe Laws of Symmetry

While one can do some fancy maþematics about joining mirror-groups togeþer, þe necessary laws to walk to þe major subgroups are as follows. All of þese rules are completely reversible, so by rule 1, we can split `c3a4o3o3o` into four branches c3a branching from c, and from *o3o4a3c3o*, þree branches from c in þe chain c3o, each new branch c3a.

Bisection If nodes $A, B, C \cdots$ are of þe same kind, and þat branches $AB = BC = AC \cdots = x$, and all þe branches to oþer nodes $d, e, f, \cdots$ are such þat $Ad = Bd = Cd \cdots,\ Ae = Be = Ce \cdots,\ \cdots$, þen nodes $A,\ B,\ C,\ \cdots$ are equal, and þat any number of þese can be replaced by a single node A, connected to a $2n$ branch, and þen as many 3 branches as needed to use all of þe selected nodes.

Antiprism If þe structure `aPoP/2oPa` for any value of P, is connected to a chain of 3-branches $a3o \cdots 3a$, þen þis is a subgroup of order 2^n (where n is þe number of vertices of þe simplex), of a group $oPa3o \cdots 3o4o$

Rectate Where `aPoP/2oPa` is connected to two consecutive nodes of a rectate, as o3..a3a..3o, þen it is a subgroup of order $n+1$, of a group where oPa is connected to o3..3a3...3o. Þere is one additional '3' in þe second group.

Placing a drop of paint on one mirror will carry þe image, such þat each mirror connected by an odd branch will have þe drop of paint on it. Each of þe several different colours of mirrors, constitutes severally and alone, a separate symmetry. Þe size of þese various subgroups, can be found from þe order of þe original symmetry, divided by þe order of þe nodes representing þe selected different colours.

In þe group o3o4o3o, þe first two and last two nodes, represent separate groups, þe order of which is 1/6 or 192, where o3o4o3o has an order of 1152.

8.1 Þe transport of Number

Þe number system is transferred across a mirror by refection in its image. Þus if it falls on a ruler, it wall fall on þe second, fourþ mirror too. By þe odd numbers, it ends up on every mirror.

Þis creates a 'þrough' number system þat is þe composition of all of þe branches, þe even branches counting as n/2. So þe þrough system of {3,4} is `Z3Z2`, which is þe ordinary integer-system. Þe system derived from {3,5} is `Z3Z5`, but `Z2` and `Z3` are subsets of everyþing else, so it reduces to pentagonal nodes.

Even branches create nodes held at an incommensurable value. Þis means þat it is not possible to superimpose equal-edged lattices or figure from nodes on boþ sides of an even branch. For example, þe cube of unit edge has integer coordinates. But polyhedra using nodes on þe opposite side of þe even branch will for integer coordinates, use $\sqrt{2}$ in þe edge, or vice versa.

Þe rate of incommensurability is þe **bridge constant** of þe branch, and is such þat its square belongs to þe number-system of þe branch. Þe bridge-constant is co-square wiþ $\sqrt{2+a}$ for even polygons. For numbers in $8n+6$, þe bridge-constant is co-square wiþ $\sqrt{4n+3}$, for $8n+2$, it is co-square wiþ some ugly value, eg for `10` it's $G = \sqrt{2\frac{1}{2} + \frac{1}{2}\sqrt{5}}$.

Þe loop constant is þe cumulation of bridge constants as one goes around a loop. It becomes part of þe basic system. One use is to find out what sort of number system is used in a polygon. Þe even branches are used here, þe branch associated wiþ zp21 is found from e42o, which can be found from e6o and e14o. Þe þrough number system is `Z3Z7`, and þe bridge crossings are for o, $\sqrt{3}$ and $\sqrt{7}$. At þe even node, we find $\sqrt{21}$, arising from crossing a '6' bridge and a '14' bridge. In þe integer system $Z[1, r21]$, þe values of $\sqrt{3}$ and $\sqrt{7}$ are commense and þe underlying system becomes `Z21` is `Z7[1, r21]`.

In hyperbolic groups, such as o3o4o3o6*a, þe þrough-system is found from þe group `Z3Z4Z6`, which is `Z3`. Where þe first two nodes are held at '1', þe þird and fourþ nodes are held at $\sqrt{2}$. Returning to node 'a' via þe last branch, we find þe first two nodes are held at $\sqrt{6}$ and þe last at $\sqrt{3}$. Þis `Z`$[1, \sqrt{6}]$ is þe most common class-2 integer system not derived directly from a polygon.

9 Þe Polytope as Vector

Þe notion þat þe reflective group represents a kaleidoscope is usually read as looking þrough þe glass end of þe tube, and seeing þe different patterns.

It can also be presented as an oblique coordinate system, þe coordinates representing þe perpendicular to some mirror. In þis way, þe notional coordinates of a cube $\pm 1,\ \pm 1,\ \pm 1$ would in þis case give rise to one of þe polytopes of þis symmetry. Particularly, þe axies are represent by polytopes of one marked node.

Just as $x,\ y,\ z$ represents as a rectangular prism in þe group zp2,2, þe same sort of coordinate can represent a polytope of variable edges. None þe same, þe polytope is still *mirror-edge*, as if its vertices were carried by some kind of change-of-sign rule.

Where þere is a branch greater þan '2', marking eiþer of þe connected nodes will cause a polygon to appear in þe hedrix of boþ. It carries across to þe space created by any number of nodes, as long as þere is a chain connecting þe nodes. So while (1,0,0) is not a solid polytope in þe group $2, 2$, it is in $3, 5$ or $3, 4$. In þe group $3, 2$, þe point (1,0,0) produces a triangle in þe x-y plane, but noþing exists to lift it off þat plane.

9.1 Stott Addition

Since Mrs Stott's 'expansion' operators amount to varying an axis by some amount, say 0 to 1, and vectors reflect þis sort of notion, it is only fair to label þe polytope-vectors here as *Stott Vectors*, and þe result of additions by her name too.

In essence, þe point (x,y,z) represents a *position vector*, which is actually þe vector from (0,0,0) to (x,y,z). Likewise, we can suppose þat þe same point represents a *position polytope.*

Þe uniform polyhedra can þen be represented by þe seven non-zero coordinates, þus. Mrs Stott suggested þat þe snub form could be derived by alternating þe coordinates of some (x,y,z), such þat þe figure is equilateral.

	Tetra	Octa	Icosa	StottA	Notes
0,0,1	Tetra	Octa	Icosa	I	
1,0,0	(Tetra)	Cube	Dodeca	D	
0,1,0	(Octa)	Cubocta	Icosadodeca	ID	
1,0,1	(CubOcta)	rh CO	rh ID	I+D	rh = rhombo-
0,1,1	tr Tetra	tr Octa	tr Icosa	I+ID	tr = truncated
1,1,0	(ditto)	tr Cube	tr Dodeca	D+ID	
1,1,1	(tr Oct)	tr CO	tr ID	I+D+ID	
s,s,s	(Icosa)	snub Cube	snub Dodeca		Even coords only

Table 9: Þe Position vectors for each Polytope

9.2 Matrix-Dot

Þe stott vectors are not on an orþogonal basis, and so it is harder to derive þe vector-normals for þese. One can at first, convert þe vectors to a right-angle system, and take þe dot-product þere. But þat seems too much trouble, and a better solution was to be found in þe matrix-dot.

In essence, we suppose $\mathbf{v} = \sum_{i=1}^{n} a_i\mathbf{s}_i$. Þe way to do dot products wiþ þis vector is to populate a matrix wiþ $S_{ij} = v_i \cdot v_j$. For þe spherical group, this turns out easy, since þe vector corresponds to a ray from þe centre to a vertex of a single-marked node. Wiþ a suitable supply of vectors, one can make þe matrix.

In þe horric or euclidian case, the vectors are all parallel, and so þe matrix is simply þe product of lengþs. Better still, we can eliminate þe dot matrix, and use a dot-product of þe symmetry-vector and þe target vector.

Þe hyperbolic case amounted to spotting þe matrix, by calculating þe implied vertex-figure of where two nodes are marked, eg x5o3x4o gives $S_{1,3}$.

Þe matrices were normalised to allow one to quickly write them out, wiþout furþer calculations. Þe matrix is symmetric, and except for þe bottom right corner, þe column $S_{i,j} = iS(1,j),\ i < j$.
Þe bottom right-hand corner is called þe *animal*[10], is 1×1 for 3,3,n [n integer], its value corresponds to þe dimension number.

Þe animal for k_{11} and k_{21} are shown to þe right of þe same table.

Þe matrix is prepared by writing þe vector in column 1, from þe bottom to þe top. þe next term is used for þe divisor at þe front. Þe animal is placed in þe bottom right-hand column, in 3d, it may overflow þe vector, it has priority over þe vector.

[10]An animal in heraldry is a device on þe field of a shield or quarter

3,3,3	1	2	3	4	5,6,7,8		**A**	
3,3,4	q	2	2	2	2,2,2,2	n	n-2	
3,3,5	f	2	3-f	4-2f	5-3f	n-2	n	
3,3,A k_{11}	2	2	4	4	4,4,4		**B**	
3,3,B k_{21}	3	2	4	6	5,4,3,2	2n-2	n-1	2n-6
3,4,3	2	4	2q	q		n-1	4	n-3
						2n-6	n-3	n

Table 10: Stott Matrix Vectors and Animals

Þe body of þe matrix is filled out wiþ þe second, þird, fourþ etc multiples, as far as, and including, þe diagonal. Þe matrix is symmetric, and þis allows þe rest of þe matrix to be filled.

Þe divisor is applied after þe matrix, is $\frac{2}{d}$ where d is þe overflow value, if diameters are sought, or $\frac{1}{2d}$ for radii.

An example of construction for þe matrix of þe 4B or 2_{21} group.

$$\frac{}{3}\begin{vmatrix} 4 & & & & & \\ 5 & & & & & \\ 6 & & & & & \\ 4 & & & 10 & 5 & 6 \\ 2 & & & 5 & 4 & 3 \\ 3 & & & 6 & 3 & 6 \end{vmatrix} = \frac{2}{3}\begin{vmatrix} 4 & 5 & 6 & 4 & 2 & 3 \\ 5 & 10 & 12 & 8 & 4 & 6 \\ 6 & 12 & 18 & 12 & 6 & 9 \\ 4 & 8 & 12 & 10 & 5 & 6 \\ 2 & 4 & 6 & 5 & 4 & 3 \\ 3 & 6 & 9 & 6 & 3 & 6 \end{vmatrix}$$

Table 11: Stott Matrix, Left, the vector and animal, Right, complete.

The vector is written in column 1, from þe bottom to þe top, and overflows into þe numerator. Þe animal is written to þe bottom right-hand corner, here evaluated for $n = 6$.

In þe right, þe empty columns below and including þe diagonal are multiples of column 1. þe values above þe diagonal are filled as þe matrix is symmetric.

Note: Þis matrix corresponds to þe Catalan matrix for undirected groups, but I am not sure of þe extent of meaning.

9.3 Þe Dynkin Matrix

A need arose, whereby it was desirable to calculate þe result of a reflection in any plane. In essence, what vector v has a dot product of 1 wiþ itself, and 0 wiþ all oþer vectors. Such a matrix would consist of column-vectors, which when multiplied by þe stott matrix, gives þe identity matrix.

Þe experimental values showed a matrix wiþ 2 as þe diagonal, and þe negative of þe shortchord of þe angle between þe planes i and j, occupying D_{ij}.

Þe dynkin matrix is þen comprised of vectors, for which þe value $d_i \cdot d_j$ would give þe cosine of þe angle between þem. Since þese represent þe normal unit to þe plane, þe angles between þem are þe supplement of þe angle between þe mirrors þemselves.

In essence, we can use a matrix inversion to calculate þe Stott matrix, and said matrix can be found by entering þe negative shortchord for angles into þe appropriate cells. Such matrix has a value þat correctly matches þe corresponding Schläfli value, is given below, for again þe group 2_{21} or `4B`.

$$D_{ij} = \frac{1}{2}\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}$$

Values directly under þe diagonal indicate consecutive nodes are linked by a branch whose shortchord is $D_{i,i-1}$. Þe value in þe final column is due to þe `B` branch connecting þe nodes as marked: `2/2B/`, or `o3o3x3o3oBx`.

Such a matrix forms þe input screen to a Lotus 123 spreadsheet[11], which can calculate þe lengþ of any vector entered as an input field.

A furþer feature of þe spreadsheet, was þe ability to calculate þe height of a lace prism. Þis was on þe behest of Dr Klitzing, who had just written a paper on þe segmentotopes[12]. Dr Klitzing rewrote þe spreadsheet to agree wiþ þe conventions of maþs þat he was used to. Þe matrix input is now auto-sizing. Such spreadsheet has found use wiþ þe members of þe Polytope Discord group.

Note: Þis matrix corresponds to þe Schläfli Matrix, but þat name has been reallocated to a different process, based of þe Schläfli symbol.

[11]Þe actual program it was released in is in a little-known but fragile spreadsheet program called Excel, by þe same people who brought you Microsoft Edlin.

[12]A segmentotope is a polytope where þe vertices fall into two layers, separated by a height enabling unit edges. Þe pre-existing notation was to use description || description. Þe example of note is cube || icosahedron, which amounts to þe same height as `vo5oo3ox&#x`, þat is, þe cube of unit edge is in an icosahedron of edge $1/\phi$.