

The mathematics behind polytope theory

Wendy Krieger
wykrieger@bigpond.com

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Abstract

A derivation of polytope theory beginning with number theory and bases. With the derivation comes the logic behind the

1 Introduction

The study of polytopes begins with base arithmetic. There are a number of rather unexpected connections between bases and polytopes, principally arising from the cyclotomic numbers. The notation advances with some rather interesting views.

The mathematics is not itself complex, but rather it is the unstated assumptions that sets this from conventional. Somewhere, religious beliefs creep into the process.

Mathematicians design their art as to provide a quick path to the matter in question, using ordinary words for this. The question is not so much to create a unified naming, but more to lay duck-boards to the matter in hand. Often the beauty of the situation is lost.

1.1 Naming Conventions

I find that the names used in this subject lead to confusion and a good deal of wasted time. If as much care had been taken to giving names as to the mathematics, this problem might not arise. So in the style of Oliver Heaviside, I shall use a *rational* system of names, and set the text accordingly.

Names given in honour of people, serve to confuse and mislead later researchers. The so-called “Wythoff Notation” is an example, which might lead people to seek out Wythoff’s papers, or suggest that it’s “Wythoff’s Notation”. Both of these things I have done in front of one of the authors of that notation.

Another source of confusion, is the use of different people’s names to designate part of the same process. Stott’s vectors are by common names, to be normalised using Gram matrices, and Dynkin’s symbols rendered into Schläfli Matrices, even though some of these people have more important roles here.

Schläfli is the first of many to describe regular polytopes by their sillage, that is, a series of numbers representing how many around an $N-2$ element. There are useful processes that come from these, discussed later, but none involve the so-called Schläfli matrix. That is actually a representation of the unmarked Coxeter-Dynkin symbol in matrix-form.

In the rational notation, we shall extend a single name across an entire process, so one is aware that this goes with that.

When the dimensionality of something is increased, one must consider if the new element is a ‘+’ or an ‘=’ role. Where a name is taken to preserve a constant number of ‘=’ elements, it means the term is held relative to solid. For example, *surface* is taken to represent a single $x = 1$ style operator. This is a partition of space, and corresponds to a dimensionality of $N-1$.

Common usage is to preserve the number of ‘=’ signs in changes of name, while the mathematicians preserve the number of ‘+’ signs. A face in three dimensions is ‘++=’, that is two dimensions dividing space, but in four dimensions, it is used to represent ‘++==’, two dimensions, while a new term *facet* is used to describe ‘=’, eg ‘+++ =’, being three dimensions dividing space. In four dimensions, ‘a tesseract has eight facets, each of which has six faces’, is by no means absurd, when facet simply means a small face. The rational way is to say ‘A tesseract has eight faces, each having six margins’.

It may not be such a problem, until you recall that other words have to be re-purposed or invented to fill the gap. Norman Johnson was using 'cell' in the sense of a 3d surtope, (choron). When I suggested what would he do with the word in 'cellular', he suggested 'cellule'. Such was the effects at play that this is now the 'standard term' for what people call 'cell'.

Equally offensive is 'weight' to designate the force of weight, when it correctly designates what is called 'mass' (measure). The acts specify that weighing shall take place on a balance, so the ruling equation is that of a torque balance, viz, $mgL = MGL$. The weighing is correct when both pans swing, or weigh. Given the size of the balance is such that the error in $g - G$ is significantly less than other errors, we can suppose this. The act specifies $L = l$ in the primary instance, so the weights are identical, be it in London or Darwin, or even the moon. The balance does not work in free space, which is why one is weightless. Spring scales, which correctly measure the force of weight, have to be adjusted to indicate the true weight.

Yet this does not stop large numbers of vandalisations of the correct usage on places like the Wikipedia, where students have been told by their professors that weight is a force, and so forth. A new word is needed, and *heft* is the rational choice.

2 Bases

By the enthusiast, a base represents a replacement for the current decimal system. The current state of this art is the modern decimal implementation: one has a digit for each column, and the arithmetic is implemented by tables. It is this second element which limits the choice of base. In essence, one might have to learn the arithmetic tables for b^2 separate elements.

Bases lead to measurement systems, the general rule is to clone the decimal metric system, fixing up perceived errors. Attaching a measurement system brings into play much larger numbers. While the ordinary count might bring 100 or so to mind, measurements of length range from millimetres to kilometres, often something like nine places of decimal.

For the count, a number is grouped into a number of batches of size b , and a remainder. This continues until all one is left with is remainders. Each count of remainder has its own symbol: a digit. Older number systems might have symbols to represent the order of count, repeated for as many times as the remainder. Thus 12 might become XII, meaning a batch of 10, and two remaining.

The method of converting between bases, is to replicate this casting of groups, the division is a repetition of the base, the outcome appears in the remainders. Thus, to convert 1000 to dozenal, one notes it is 83 dozen, and 4 remainder. The count of dozens is a number, and groups to 6 dozens remainder 11. Then 6 groups to nothing and a remainder of 6. The number is then 6.11.4. A symbol for the additional digits V for decimal 10, and E for 11, allows us to write this as a fairly ordinary number 6E4.

There is little mathematics in this activity. It is more a linguistic enterprise, the issue of the day is to name the extra digits, and what names the columns ought have. The example of 6E4 might be 'six hundred and eighty-four', or 'six gross, eleven dozen and four' or any of a range of issues. Part of the question is is the new base to be by itself, or is it going to live beside decimal.

Bases are taught in middle-grades, such as to children at the age of ten or eleven. It is a short session. In Pendlebury's *Shilling Arithmetic*, it occupies a half of a page. On the other hand, the decimal fractions occupy ten or twelve pages of text.

Decimal fractions are derived by successive multiplication of the fraction, the integer being the remainder, and the new fraction ensures. Unlike the count, this process usually has no end, instead, one stops when sufficient digits are found. Thus an English foot, rendered into metres goes 10 feet make 3 metres and a bit. 10 bits makes no metres and a bit. 10 bits make 4 metres and another bit. 10 bits make eight metres.

In the work of Stevin's 'La Disma'. This fraction would be 3 primes, 4 birds and 8 fourths. The decimal or unit point would take another forty years after this work. Converting the fraction is to simply replicate this sequence of remainders, but with 12 as the multiplier.

2.1 Periods

One of the common activities that base enthusiasts like to do is to prepare a table of reciprocals of small numbers in several different bases, as if such might attract new users. Certain numbers have a terminating reciprocal, that is, they divide some power of the base. The great majority have a continuing fraction. For

example, in decimal, $1/8 = 0.125$ while $1/9 = 0.111111\dots$. Some reciprocals enter a period after a leader, so $1/6 = 0.166666$, the '1' bit is a leader, and the '6' bit is the same as in two-birds, $\frac{1}{6} = \frac{2}{3} - \frac{1}{2}$.

Since from the right of any point in the period, the digits represent some numerator on the same denominator (for example, in $1/7 = 0.142857\dots$, the digits beginning at the '5' represents $4/5$), the pattern recurs at some number less than the denominator.

Given that $ax = b \pmod{b}$ and that the periods of every numerator is the same length for primes, the period must divide $p - 1$. This is Fermat's little theorem. Gauss extended this by a test as to whether there are an even or odd number of loops. This is quadratic reciprocity.

2.2 Algebraic Roots

The algebraic roots are the real factors of $b^n - 1$. These exist for each divisor of n . One can find these from the decimal values, using a BIGNUM routine, the following works for roots as many as 163 digits.

The odd numbers have fewer divisors, so we follow the Cunningham Project order, of placing the odd numbers one step before the corresponding double.

| o | e | decimal | +5's | equation |
|----|----|-----------|------------|---------------------------------------|
| 1 | | 9 | 564 | $x - 1$ |
| | 2 | 11 | 566 | $x + 1$ |
| | 4 | 101 | 5656 | $x^2 + 1$ |
| 3 | | 111 | 5666 | $x^2 + x + 1$ |
| | 6 | 91 | 5646 | $x^2 - x + 1$ |
| | 8 | 10001 | 565556 | $x^4 + 1$ |
| 5 | | 11111 | 566666 | $x^4 + x^3 + x^2 + x + 1$ |
| | 10 | 9091 | 564646 | $x^4 - x^3 + x^2 - x + 1$ |
| | 12 | 9901 | 565456 | |
| 7 | | 111111 | 5666666 | $x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$ |
| | 14 | 909091 | 5646464 | $x^6 - x^5 + x^4 - x^3 + x^2 - x + 1$ |
| | 16 | 10000001 | 56555556 | $x^8 + 1$ |
| 9 | | 1001001 | 5655656 | $x^6 + x^3 + 1$ |
| | 18 | 999001 | 5655456 | $x^6 - x^3 + 1$ |
| | 20 | 99009901 | 5654565456 | $x^8 - x^6 + x^4 - x^2 + 1$ |
| | 24 | 99990001 | 565554556 | $x^8 - x^4 + 1$ |
| 15 | | 90090991 | 5645646546 | $x^8 - x^7 + x^5 - x^4 + x^3 - x + 1$ |
| | 30 | 109889011 | 5665444566 | $x^8 + x^7 - x^5 - x^4 - x^3 + x + 1$ |

Table 1: Small Algebraic Roots

Adding a string of 5's turns the decimal number into a form with negative digits in the base. The polynomial is then derived by replacing the digit in 10^x by $d - 5$. At 163 digits, there are a few that require ± 2 as a polynomial coefficient, this process correctly handles this.

Primes that have a period of n will divide the entry corresponding to n in this table. This entry is denoted as bAn . The list of primes that divide bAn forms the 'Yates' table¹. The balance of numbers not dividing these are the *repeaters* and *sevenites*. Where p divides some bAj , then it also divides some $bApj$. For example, 3 divides $10A1$, and so it divides $10A3$ and $10A9$.

A much rarer occurrence is where p^2 divides some bAn . These are the sevenites, which for any given base is extremely rare. The name refers to the smallest compound sevenite, where $7^3 \mid 18A3$. Decimal has three known sevenites, being 3, 48756598313 run as far as 120^4 . It is extremely rare to find a large compound sevenite (ie $p^3 \mid bAj$, $p > b$), only one is known under $b < 2,000,000$ ($b=68$, $p=113$).

¹After the corresponding table in Yates' book *Repnits*.

2.3 Pe base as $b^n - a^n$

2.4 Alternating Bases

It would be remiss not to discuss the historical number systems. These are not cosy numbers near ten, but rather large numbers, multiples and fractions of twenty are the order of the day, but other systems appear sparingly. Many of these systems do not use a single value per column, but alternate. The same logic applies as before, except now one must care which number produces the remainder.

Bases 18 and 20 are large enough to use a two-row abacus, and one alternates between numbers that make these. A score is made of tallies, each of units. A tally of scores might make a block. We should show the count of 118 in decimal, in both of these systems.

In base 18, a tally is 6, and a score is 18. The 118 coins make 19 tallies of 6, and four left. 19 is then bundled into 6 scores and 3. A group of 6 scores make a block. We might write this as ' '4, meaning a tally of scores, a tally and four.

In base 20, a tally is 5, and a score is 20. The 118 coins make 22 tallies and three, the 22 tallies give 5 scores and 2, and the 5 makes a tally of scores. We get ' '3.

Conversion can be done directly, since each uses a series of columns. ' '3 divides by six to give ' '4 remainder 4. Dividing this lot gives '1 remainder ', and '1 gives a block of 6: ' remainder 0. So ' '3 (b20) is ' '4 (b18).

These alternating systems derive in part from having two rows on the count-table, with different widths, a count in the lower row overflows to the upper row, and the upper row overflows to the next column.

2.5 Pe Base as Integer System

The set B_n describes numbers that can be written exactly in base n . While most of the primes still remain primes, the erstwhile primes that divide n become units of the system. For example, in the set B_{10} , the numbers 2 and 5 have terminating reciprocals, and thus $\frac{1}{2}$ and $\frac{1}{5}$ can be written exactly in this system.

Base enthusiasts make a good deal of the versatility of these numbers.

If one supposes that the fractional part of a number is in base $\frac{1}{b}$, then there is a direct correspondence of the numbers between 0 and 10, and the integers, by the simple ruse of reversing the index. 12345, for example, corresponds to 5.4321. This closely represents the correct way to calculate the size of the fractional part, since it often meets the line of all a/N is represented for $a = 1$ to N , at powers of any size.

This means that while any binary number can be exactly represented in decimal, this does not directly confer to rulers. The decimal $1/1000$ contains all of the eightths, but the binary ruler of this order would go to $1/1024$. The order of intersection is thus calculated as a *cascade*. Decimal numbers contain binary numbers, at the same rate that they contain numbers made of binary digits (0, 1), but allow for ten digits. So it corresponds to $\ln 2 / \ln 10 = 0.30103$. A ruler divided into 1000 parts, contains $1000^{0.30103}$ parts, or 8 in total.

Where the bases contains several primes, the proper cascade is found by adding the least portions for each. For example, with 120 and 10, the primes are 2, 3, 5, in 120 we find that there is actually 8 for 2, so we get

$$J = \min\left(\frac{\ln 8}{\ln 120}, \frac{\ln 2}{\ln 10}\right) + \min\left(\frac{\ln 3}{\ln 120}, \frac{\ln 1}{\ln 10}\right) + \min\left(\frac{\ln 5}{\ln 120}, \frac{\ln 5}{\ln 10}\right) \\ = \min(0.43438, 0.30103) + \min(0.229475, 0) + \min(0.3361756, 69897) = 0.63720555$$

This produces an intersection of 81.583 for 1000 in decimal, or 21.1278 for 120, which is different to the cumulation of 2-5 numbers. This is because the decimal digits is smaller than the twelfth, and more digits are needed to show the same large N .

3 Chord Arithmetic

The element in chordal arithmetic, is to take a step, and veer $2a$ to the right, and then take another step of the same size. The bisector to the right is at angles b to the first step, and this in general produces a series of isosceles triangles of angles $2a, bb$. The ray connecting the beginning of the first step to the end of the second makes an angle to each step equal to a .

$$\begin{array}{cccc}
i & ii & iii & iv \\
t(n+m-1) & = a(m-1)t(n) & -t(n-m+1) & a^{\wedge}m-1 \\
t(n+m) & = a(m)t(n) & -t(n-m) & a^{\wedge}m \\
t(n+m+1) & = a(m+1)t(n) & -t(n-m-1) & a^{\wedge}m+1
\end{array}$$

Since columns *i* and *iii* represent isoseries in *a*, so must column *ii*, which is the sum. Dividing through by $t(n)$, the constant at each step is an isoseries of *a*. It requires little further to puzzle out that $a(0) = 2$ and $a(1) = a$. The result behaves like a power-series.

3.2 The Isoladder

The isoladder is an algorithm which allows one to find arbitrarily large members of an isoseries, in logarithmic time. It runs at $\frac{2}{3}$ of the speed that is used to find ordinary large powers.

In the following example, it is desired to find the 37th member of a series beginning with terms t_0, t_1 using a shorthand s_1 .

| io | i1 | s2 | t0 | t1 | t2 | t3 |
|----|----|----|----|----|----|----|
| 37 | 1 | 1 | 0 | 1 | 2 | 3 |
| 18 | 0 | 2 | 1 | 3 | 5 | - |
| 9 | 1 | 4 | 1 | 5 | 9 | 13 |
| 4 | 0 | 8 | 5 | 13 | 21 | - |
| 2 | 0 | 16 | 5 | 12 | 37 | - |
| 1 | 1 | 32 | 5 | 37 | | |

Table 2: Calculating s_2^{i1} from a series beginning t_0, t_1

The values i_1 and i_2 are integers, the registers needed for these need not be large. The variables $s_2, t_0 - t_3$, are done in a bignum process. This could mean doing a modulus step at each calculation.

The algorithm runs along these lines. This is REXX² code that runs this. For odd i_0 , t_0 and t_1 end up with t_1 and t_3 . For even i_0 , t_0 and t_1 become t_0 and t_2 . The % is integer division, the // is integer modulus.

```

isoquad: procedure;
parse arg s2, io, to, t1
if a=0 then do; t1=t0; io=1; end
if a<0 then do; io = -io; t1 = to*s2-t1; end
do forever; if io = 1 then leave
i1 = ao // 2 ; t2 = t1*s2 - to
if i1 = 1 then t1 = t2
else do; to = t1; t1 = t2*s2-to; end
io = io % 2; s2 = s2 * s2 - 2; end
return t1

```

3.3 Isobases

The isopowers correspond to a pair of algebraic roots $b + a = n$, and $ba = 1$.

3.4 The Pentagon

The pentagon answers to the equation $x^2 - x - 1 = 0$, which we can implement as a stone-table, where coins in two adjacent columns, become one in the column to the left, viz $11 = 100$ (in the same base).

²ReginaREXX is available for most platforms. The code is SAA REXX

3.5 Pe Hypercomplex plane

Pis is graphic representation of a system where $j^2 = 1$, in much pe same style as pe complex plane. Pe various class-2 systems, like pe pentagonal, octagonal and dodecagonal systems fall on it.

It acquires pe hyper- prefix, by virtue pat trig functions on pe ordinary complex plane become hyperbolic trig functions here. All of pe functions are replicated, except now pings like $\cos + i \sin$ become $\cosh + j \sinh$, and circles of given radii become replaced wip hyperbola of given differences.

Pere are unit hyperbola, and a corresponding -1 hyperbola. Pe units of any system lie on pese hyperbolae. A rotation in pe plane becomes a hyper-rotation, pat is an area-preserving skewing of pe shape along pe hyperbola.

Also present are pe zero and alt-zero axes, which represent pe parts of pe o-hyperbola. Pe perpendicular distance from pese two axes, represent pe real and alt-real size of pings. An octagon in pe real space becomes by inversion, an alt-octagon.

Numbers pen have two signs, one real, and one alt-real. Pe distances observed in finite polytopes have squares whose sign and alt-sign are both positive. Pe ones wip +- signs have an piecewise finite³, and a sparse⁴ Furber, because pere is an isomorph, by replacing j wip $-j$, pe points are restricted to a finite area where bob pe real and alt-real squares fall in a definite ranges.

3.6 Pe Heptagon

4 Pe Radiant Solid

A solid is taken to be a region of space, occupied by the substance of pe figure, for which pere is a definite boundary, which wipin it, it is present, and wipout it, it is absent. One can consider obvious non-solid distributions, such as pe Gaussian dot, answering to $d = \exp(-\text{rss}(x, y, z))$

Pe prototype solid is taken to be a double-unit sphere⁵ which answers to pe equation $r^2 = x^2 + y^2 + z^2$, or $r = \text{rss}(x, y, z)$. Here **rss** stands for root-sum-square. Such a figure has a clear surface at 1, values less pan pis are interior, pose greater are exterior.

Pe solid is taken as a function of surface. Specifically, where a ray crosses pe surface a number of times, each instance is found separately. So, pe sphere is not taken as a position of angle $\text{fn}(r, \theta)$ but of surface, pat is, $\text{fn}(r, S)$. Pe integrals are over surface, not over angle.

Pe surface is taken as $n = \nabla d$, pat is, pe gradient of pe density vector. Because pe solid has no motion, $\oint dS$.

Pe enclosed volume is pe moment of surface, pat is $v = \oint \nabla x \cdot dS$. Because pe volume is independent of where pe moment is measured from, it follows pat $\oint \mathbf{n} \cdot d\mathbf{S} = 0$

If pe surface were to be broken into two parts, pe ring⁶ separating pe two parts span a definite vector area, and pat is independent of pe closing surface. Pe sum of pe two halves must always be 0.

Pe radiant function, pen for each value of x , defines a copy of pe solid at pe scale x , where 1 is full-sized. When x is set to $\frac{1}{2}$, pen pe result is a half-sized figure.

4.1 Pe radiant space

For radiant space is one where pe point (x, y, z) , represents a copy of pe cartesian product of pe solids in pese spaces, at pe radiant function. pe solids we draw in pis space is formed by pe integration of points. Each of pese axes can represent separate solids of any dimension.

Pe generic prism product is represented by pe cube wip pe diagonal $(0,0,0)$ to $(1,1,1)$. Pe radiant function is $\max(x, y, z)$, where pe maximum is over pe absolute value. Pis corresponds to cutting out of pe X, Y and Z spaces, pe shape so represented. Pe word *prisma* is as off-cut.

A cylinder can be represented as a stack of coins, each coin is replicating pe bottom base at pat height. It can also be represented as a faggot of matches, where each match represents pe full height and a point on pe base.

³Piecewise-finite means pat it is possible to construct pe incident surtopes on any given surtope. An example is $\{\frac{5}{2}, 4\}$.

⁴Sparse here means pat every point of space is no more pan some finite x from a vertex, and no two vertices are closer pan y from each ope.

⁵In real life, pe circles and spheres are measured by peir diameter. Radius is only used for small arcs of a circle, such as deviations and range.

⁶A ring is a closed boundary on pe surface.

The radiant function $\text{sum}(x, y, z)$ represents a different product. The product applied over unit edges gives the orbotope, with the canonical vertices $(\pm 1, 0, 0)$. This product covers the surfaces of the various base figures, in much the same way that a tent covers its pegs. Originally, the word for tent was put forward, but *tabernacle* is already taken. Instead, a word for *cover* was chosen, this being assimilated to **tegum**.

Unlike the prism product, the tegum is a *drawn* product, or one of *draught*. The allusion here is to gum might draw into strands as the two parts are separated. The interior does not take part.

The pyramid product was found from the surface of the tegum. It corresponds to the plane $X + Y + Z = 1$, which draws the surtopes at the corners into a progression of prisms, the size of the prisms being unity.

4.2 Coherent Products

In a measurement system, the coherent units of space are the prism product of the length measure. Where different products used, different coherent units may arise.

For the space described above, one can start with a prism or tegum, and replace any axis with a different shape of the same dimension. Since this will cause each layer of the product to increase proportionally to the old volume to the new volume, the volume of the product is the product of the volume. To this end, the sphere, cube and octahedron has been so tested, and lead to separate coherent volumes. Note that the volumes are not equal, the octahedral line is only 1/6 of the cubic line.

| | Pn | Cn/Pn | Cn | Cn/Th | Th | Cn/Th |
|---|-----------|------------------|-----------|-------|------------|----------------------|
| 1 | linear | 1 | diametric | 1 | diagonal | 1 |
| 2 | square | $4/\pi$ | circular | 2 | rhombic | $2/\pi$ |
| 3 | cubic | $6/\pi$ | spheric | 6 | octahedral | $1/\pi$ |
| 4 | biquadric | $32/\pi^2$ | glomic | 24 | tetrtegmic | $3/4\pi^2$ |
| n | Prismic | $\eta^{n/2}/n!!$ | Crind | $n!$ | Tegmic | $\eta^{n/2}/(n-1)!!$ |

Table 3: The Coherent Products

It is then possible to write a new physics, using these measures, especially the tegmic units, as base units. The measurement of volume from area becomes $V = \int \mathbf{r} \cdot d\mathbf{S}$, and one might need to reevaluate the relation of inverse-square laws. For example, the rational form of the coulomb equation becomes $F = \frac{cQ_1Q_2}{8\pi r^2}$, the extra factor of 2 comes from the surface of a sphere is now $S = 8\pi r^2$.

4.3 Altitude

Altitude refers to those axes over which draught and copy happen, like that of the antiprism or pyramids. An antiprism is the drawing of a figure onto a parallel copy of its dual.

If one supposes a three-dimensional altitude, with antiprisms forming the axes, then one will note that the cover is a tegum product. None the same, any face of the octahedron-in-altitude is dual to its opposite face. and thus the tegum-product of antiprisms is itself an antiprism.

Likewise, the dual case, of the prism-product of antitegums (dual of antiprisms, but has its own construction), is itself an antitegum. The construction of an antitegum is to project cones over parallel dual figures, and take the intersection. But in the prism, the opposite corners of the cube-in-altitude represent the pyramid product of the figures, the opposite is the pyramid product of the dual, and hence the intersection is an antitegum (for each great diagonal of the cube).

Using large dimensions as the altitude lead to very large dimensions. A lacing of lines in this manner can lead to a five-dimensional solid quite easily. The table below shows a triplet of rectangles, parallel in one axis and orthogonal in the other, set at the corners of a triangle ABC. The result is five-dimensional, with just 12 vertices.

| | | | | |
|---|---|---|---|--------|
| A | x | x | o | square |
| B | x | o | x | square |
| C | o | x | x | square |

The pyramid product might be understood in this light, but where the bases appear only once along the main diagonal.

4.4 Surface, Periform, Hull

The boundaries of a solid vary when the surface is let cross itself. The example here refers to the small stellated dodecahedron $\{\frac{5}{2}, 5\}$.

The **hull** is the least convex shape that contains the points in question. The vertices of the stellated dodecahedron is an icosahedron.

The **periform** is the shape equal to the referenced points of a solid, excluding all exterior points. It is the shape that you would make in modeling a polytope. The periform is an apiculated dodecahedron, with pyramids raised on each face.

The **surface** is the gradient of density, and its vector-moment is the volume of the polytope. There can be parts of the surface that are interior to the periform, such as the pentagons in the pentagrams, which are a d2 (density-2) wall separating endocells of d1 and d3.

4.5 Endocells

A surface that crosses itself, will divide the interior into a number of different cells, each cell has its own density.

5 The Schläfli Series

If a is the short-chord of the polygon, the diameter D of the same polygon can be found from the relation $D^2 = \frac{4}{4-a^2}$. The maps is simplified by working with the squares, rather than the actual lengths. This happens if you have no way of finding the square root⁷.

In this calculation, if P is a polygon, then p is its shortchord-square. Learning the values of P and p was a way of getting around not having the necessary trig and log tables.

Where $\{P, Q\}$ is a polyhedron, then its vertex-figure is $p\{Q\}$. It follows from this that the diameter can be found by using the same formula as above. The iteration in this table is by eg $\{P, Q, R\} = 2\{Q, R\} = p\{R\}$

| | | | |
|--------|------------------------|-----------------------------------|------------------|
| S | s | $\frac{2}{4-s}$ | $\frac{2}{4-s}$ |
| RS | $\frac{4r}{4-s}$ | $\frac{4-s}{4-r-s}$ | $4-s$ |
| QRS | $\frac{q(4-s)}{4-r-s}$ | $\frac{4(4-r-s)}{16-4q-4r-4s+qs}$ | $8-2r-2s$ |
| $PQRS$ | | $32-8p-8q-84-8s+2pr+2ps+2qs$ | $16-4q-4r-4s+qs$ |

This function is symmetric, in that $f(p,q,r) = f(r,q,p)$. From this function alone, one can calculate the radius of the polytope as $f(verf)/f(figure)$. It turns out that the function of a product is the product of the function, so $f(a)f(b) = f(a \times b)$. The final column of the table gives the Schläfli function for 3, 3, ..P. The value A is for $k_{1,1}$, while B equates to k_{21} .

When this index first comes to zero, from positive values, the result is a euclidean tiling. We might note that in 3D, the condition for $16-4q-4r-4s+qs=0$ can be rewritten as $4r=(4-q)(4-s)$, it is the factor 4 on the right-hand side that severely limits the values that lead to a polyhedron with rational angles. We should bring to bear some powerful theory to finish that proof.

The values for 3..3,5 and 3..3,6 can be written as $\phi^2 - n/\phi$ and $3-n$, where the initial value represents the square of the shortchord.

5.1 Antiprisms

Antiprisms answer to the chord being the base, that is, $x^2 - 1 - p = 0$.

It is interesting to note here that the shortchord-squares include values that have a complex conjugate, that is, the usual rules of isomorphism do not apply here.

⁷In some parts of the world, calculators in the 1970s were still expensive things.

| Polygon | P | a | a^2 | Sch $\{3,\dots,P\}$ |
|-----------|---|---------------|---------------|---------------------|
| 2 | R | 0 | 0 | $2n$ |
| 5/2 | V | 0.61803398875 | 0.38196601125 | |
| 3 | S | 1 | 1 | $n + 1$ |
| 4 | Q | 1.41421356238 | 2 | 2 |
| $k_{1,1}$ | A | | 2 | 4 |
| $k_{2,1}$ | B | 1.5 | 2.25 | $9 - n$ |
| 5 | F | 1.61803398875 | 2.61803398875 | $2 - (n - 1)/\phi$ |
| 6 | H | 1.73205080757 | 3 | $3 - n$ |
| 7 | | 1.801937736 | 3.2469796037 | |
| 8 | | | 3.41421356237 | |
| 10 | | 1.90211303259 | 3.61803398875 | |
| 12 | | 1.93185165259 | 3.73205080757 | |
| | U | 2 | 4 | |

Table 4: Commonly used shortchords of polygons

| | a | $4a^2$ | Remark |
|----|---------------|---------------|-----------------|
| C2 | 1 | 1 | Tetrahedron |
| | 2.5 | 1.61803398875 | |
| C3 | 1.41421356268 | 2 | Octahedron |
| C4 | | 2.41421356238 | |
| C5 | 1.61803398875 | 2.61803398875 | Icosahedron |
| C6 | | 2.73205080757 | |
| U | 1.73205080757 | 3.00000000000 | Triangle stripe |

Table 5: $\mathfrak{p}e$ indicated shortchords for antiprisms as $\{3,P\}$

5.2 Snub polyhedra

$\mathfrak{p}e$ snub cube and snub dodecahedron can be realised as polyhedra of $\mathfrak{p}e$ type $\{3,sC\}$ and $\{3,sD\}$. Using $\mathfrak{p}e$ circumradius of $\mathfrak{p}e$ figures, and $\mathfrak{p}e$ formula $a^2 = 3 - 1/(-1 + r^2)$, where r is $\mathfrak{p}e$ circumradius, we get these indicated shortchords.

| p | circumradius | a | a^2 |
|-----|-------------------|-------------------|--------------------|
| 3,2 | 0.707106781186 | 1.41421356238 | 2.00000000000 |
| sT | 0.951056516295 | 1.61803398875 | 2.61803398875 |
| sC | 1.343713373744601 | 1.685018324889720 | 2.839286755,214160 |
| sD | 2.15583737511564 | 1.715561499697367 | 2.943151259,243881 |
| 3,6 | very large | 1.73205080757 | 3.000000000 |

Table 6: $\mathfrak{p}e$ indicated shortchords for snubs as $\{3,P\}$

5.3 $\mathfrak{p}e$ system B2Z4

$\mathfrak{p}i$ s family is suggested by $\mathfrak{p}e$ octagons and octagrams of $\mathfrak{p}e$ uniform octahedral group. In essence, one extends these figures by supposing that only some faces are accessible, and that, for example, $\mathfrak{p}e$ octagons of $\mathfrak{p}e$ truncated cube belong also to some of $\mathfrak{p}e$ faces of a regular solid with octagon faces.

Likewise, $\mathfrak{p}e$ squares of $\mathfrak{p}e$ rhombocuboctahedron might be replaced by octagons, that a polyhedron with a girp of eight octagons arise.

5.4 Pe Schläfli Function

Schläfli put forward a request or hope, þat one day, þe order of a group might be derived from þe symbol. Sixty years later, Coxeter⁸ reported no furþer enlightenment. It still has not been attended to. Instead, þe group order is found by way of þe Euler characteristic for polytopes of þat group, or where þe polytope occurs as a face or cell of a tiling.

6 Eutactic Lattices

Coxeter defines a *eutactic star* as the normals to þe mirrors, in eiþer direction, of þe mirrors of a group. Groups wiþ only odd mirror-angles or right angles, have only one star. Each separate set of nodes þat removal of even branches leaves, is a separate star.

| Name | Cox. | Curr. | Kri. | Order | Lattice | Cox. | Cur. | Kri. |
|-------------|------|-------|-------|-------------|------------|------|------|------|
| Rectangular | | r | 2^n | prismatic | | rr | | |
| Simplex | A | A | s | $(n + 1)!$ | | P | A | t |
| Halfcube | B | D | h | $2^{n-1}n!$ | qtr.cubic | Q | D | q |
| Cubic | C | B=C | hr | $2^n n!$ | semi.cubic | C | S | qr |
| | | | | cubic | R | B | qrr | |
| Polygonal | D | I | p# | $2p$ | horogon | W | I | rr |
| Hexagon | | G | ss | 12 | hexagonal | V | G | tt |
| Gosset | E | E | g | gos(n) | gosset | T | E | y |
| | F | F | hh | 1152 | | U | F | qq |
| Pentagonal | G | H | f | pen(n) | | | | v |

gos(n) is þe product of þe first n of 1,2,6,10,16,27,56,240.

pen(n) is þe product of þe first n of 2,5,12,120

Table 7: Reflection-groups in all dimensions

7 An Overview of Notations

Coxeter's book *Regular Polytopes* does not describe polytopes as vectors. Instead, þe notation is simply þe Schläfli symbol, wiþ serves until þe discussion on þe Gosset polytopes. In order to make þese text, one supposes þat þe Schläfli symbol is a representation of þe Dynkin diagram, and þat þe branches of þese groups derive from a curtail, þis: $\{3_3^3\}$. Such figure is furþer written as 0_{21} . Pe various Gosset polytopes are þen written wiþ a succession of leading '3's, eg $\{3, 3, 3_3^3\}$ for 2_{21} . Pe Elte figures can start from any end, such as $1_{2,2}$, wiþ þe exception of 1_{42} , which fails Elte's raper artificial definition.

Pe Schläfli symbol has no notion of anyþing furþer þan what is written. While it might correspond to þe branches of a Coxeter-Dynkin diagram, þe Schläfli symbol has no notion of anyþing on nodes. It is for þis reason þat þe Stott operator was included.

One can linearise þese groups, by supposing þe shortest branch begins not wiþ a '3', but a node "3, which would mean to count backwards two nodes. Alternately, one could use letters A and B to connect þe new node to þe second or þird last node. Pe polytope 2_{21} becomes 4B. Using a number to represent a string of '3's, þen means þat someþing like $\{3,3,5\}$ could not be so expressed. þe fix for þis was to denote þese by letters too: Q for '4', and F for '5'. So $\{3,3,5\}$ becomes 2F.

Pe new names for þe polytopes ought not contain superscript or subscripts, or brackets, or quotes. Þese are meant so þat one can use þe name as a subscript $R_{0,name}$ or set þe clear limit of þe name, as 'name' þat þe reader might know what is and isn't þe name. Brackets are meant, as in maþematics, to denote þe enclosed is a single object, þus in $(3 + a) * b$, þe bracketed expression is reduced to someþing þat can be multiplied onwards. Ideally, subscripts and superscripts are best avoided completely.

Pe final allocations of symbols was based on being able to describe þe second extension⁹. Exactly where Wythoff's construction fits in is not known, but it's based on a few scattered comments in 'Regular

⁸Regular Polytopes, 1947

⁹Pe first extension is þe compact hyperbolic groups, þe second is þe paracompact groups.

Complex Polytopes'. These were also to be included.

The symbols are designated as *structural* and *decorative*. Structural elements build the kaleidoscope, two diagrams with the same structural items are the same symmetry. Decorative elements create an object for the kaleidoscope to reflect.

7.1 “Wythoff” Notation

This has no connection to Anton Wythoff. It is instead, an ‘honour-name’, the main purpose appears to mislead and distract researchers. In essence, it’s a decorated Schwarz-triangle, with the mirrors bisecting edges appearing before a vertical bar ‘|’. It is used in Magnus Wenninger’s *Polyhedra Models*.

7.2 Stott-Schläfli Notation

The more common notation, and one that works in higher dimensions, is to use the modified Stott expansion notation against a regular polytope denoted by a Schläfli symbol. The modification to Stott’s system is to start off with a zero-size regular solid, rather than a size-1 one. The regular solid is then made by expanding surtope-0, or pushing the vertices radially outwards. The raw Schläfli symbol serves as a name for the regular polytope.

Where Mrs Stott wrote e_1C_{600} , the new form becomes $t_{0,1}\{3,3,5\}$. Mrs Stott’s notation already supposes an expanded form of node 0, where the revised notation does not.

An alternate SS notation is to suppose that the individual expands run as powers of 2, from 1, 2, 4, 8..., and that the figure in question is denoted by a dimensional letter and polytope base. We have A, B, C, D... representing the dimensions from 1 upwards, and then t, o, c, i, d representing the polyhedra in 3D, or the equivalent polychoron in 4D. There is an extra 4D regular, which is given the letter q.

A figure such as the example above gives D_3 . This is a four-dimensional $\{3,5\}$, with expansions at vertices and edges. The prism-products are simply the concatenated symbols, such as a dodecahedral prism is ACd_1 . The polygons are B_p , the antiprisms are C_p . The figures are then arranged into the list according to the first instance of the polytope.

| Key | 2D | 3D | 4D | Remarks |
|-------------|-------|----------|-----------|---|
| A | B_4 | Cc_1 | Dc_1 | Measure Polytopes |
| B_n | B_n | AB_n | AAB_n | Polygon cube prisms |
| BB | | | $BpBp$ | polygon-polygon prisms |
| C | | C_p | AC_p | polygon antiprisms |
| C_{x1} | | C_{x1} | AC_{x1} | Platonic figures |
| C_{x2-7} | | C_{xn} | AC_{xn} | ME archimedean |
| C_{x8} | | C_{x8} | AC_{x8} | Snub Cube Cc_8 and Dodeca Cd_8 |
| D_{x1} | | | D_{x1} | Regular polychors |
| D_{x2-15} | | | D_{xn} | Mirror-edge archimedean |
| D_{x16} | | | D_{x16} | snub 24ch Dq_{16} and grand antiprism Di_{16} |

Table 8: Catalog of Uniform Polytopes

Some others are duplicates. Ct_4 and Ct_6 are Ct_1 and Ct_3 . Also Ct_2 , C_5 and Ct_7 are Co_1 , Co_2 , and Co_3 . Ct_8 is Ci_1 . In general, 4 and 6 are the same as 1 and 3 of the dual, so Co_6 is Cc_3 .

In 4D, the positions 4, 8, 10, 12, 13 and 14 are the same as 2, 1, 5, 6, 11 and 7 of the dual, and referred to as such. So Dq_8 is the same as Dq_1 . 6 and 9 and 15 are identical from either end, but are constructed from the base polytope, rather than the medial (6). Do_2 , Do_5 , and Do_7 are Dq_1 , Dq_2 , and Dq_5 .

7.3 Kepler names

Kepler’s names for the various archimedean polyhedra make some sense. However, they do not generalise all that well, and the words lose their meanings in some of the applied schemes. The numbers refer to the Stott-index of the previous section.

platonic (P) (1 = v) The platonic figures, by a generic face-count. These might be distinguished from other figures of the same face-count, by saying ‘regular’ P.

truncated P (3 = ve) The vertices and their verge is cut off the platonic figure, leading to a doubling of edges of the original faces.

snub (8) A twisted figure made of triangles, the non-triangular faces belong to that of the named polytope (Cc8 and Cd8). The same pattern is followed in four dimensions, with the Dq16 'snub 24choron'. Gosset provided this name.

middle (2 = e) A figure derived from bisecting the edges of platonic figures. They fall in pairs, so Cuboctahedron, Icosadodecahedron.

rhombo-M (5 = vh) A notional intersection between a middle-figure and its dual. The dual has rhombic faces (rhombo-dodecahedron and rhombo-tricontahedron), which is the source of 'rhombo' here.

truncated-M (7 = veh) Notionally a truncation of the middle-figure, except that the proper truncation would have rectangles, rather than squares. Also *rhombo-truncated*.

These names are fine in three dimensions, but in four dimensions and higher, things come a little undone. Little more than the truncate survives unchanged.

The antitegmal sequence is the intersection of duals, as one increases and the other decreases. Mapped on a higher dimension, these intersections represent slices of an antitegum of either end. The intersection produces an aggressive vertex bevel, which has the effect of moving the vertices along the edges, then when the edges are exhausted, towards the centres of the hedra, until exhaustion, and so forth, until all of the surface is worn away, and the vertices proceed towards the centre as those of the dual.

The process in three dimensions passes through truncated cube, cuboctahedron, truncated octahedron. The Cuboctahedron is middle of this series. In four dimensions, the process adds two extra steps. The edge centres of the tesseract do not align with those of its dual. Instead, the middle-point is half-way between.

The remaining two (rCO, tCO), correspond to a truncation of the cuboctahedron, until its edge centres are met. In practice, the cuboctahedron has a $1 : \sqrt{2}$ rectangle with no degrees of freedom, but topologically, the truncated and rhombo- Cuboctahedron serve as a third truncate.

base (1)(8) The base polytope, and its dual (8).

truncate (3) (12) The edges are shortened in situ, creating new faces at the vertices.

rectified (2) (4) The edge-centres of one are the hedron-centres of the other.

bitruncate (6) The middle form is now comprised of the truncates of the vertex-figure duals.

runcinate (9) The antiprism figure gives rise to this in 4D. Norman used the term to denote node 3, I use it to denote the last node.

cantellated (5) (10) This is the rectified rectate. The bicantellate is the rectate of the birectate.

cantitruncated (15) The truncate of the n-rectate gives the n-cantellate.

omnitruncated (15) This one has a vertex for each flag, and represents the extreme bevelling on every node. The dual is the variated figure.

runcitruncate (11) (13) This is not the truncate of the runcinate, or any other figure. It's not an easy row to hoe here, so the figure is just given some sort of 'fake' construction.

8 The Laws of Symmetry

While one can do some fancy mathematics about joining mirror-groups together, the necessary laws to walk to the major subgroups are as follows. All of these rules are completely reversible, so by rule 1, we can split $c3a4o3o3o$ into four branches $c3a$ branching from c , and from $o3o4a3c3o$, three branches from c in the chain $c3o$, each new branch $c3a$.

Bisection If nodes $A, B, C \dots$ are of the same kind, and that branches $AB = BC = AC \dots = x$, and all the branches to other nodes d, e, f, \dots are such that $Ad = Bd = Cd \dots$, $Ae = Be = Ce \dots$, \dots , then nodes A, B, C, \dots are equal, and that any number of these can be replaced by a single node A , connected to a $2n$ branch, and then as many 3 branches as needed to use all of the selected nodes.

Antiprism If the structure $aPoP/2oPa$ for any value of P , is connected to a chain of 3-branches $a3o \dots 3a$, then this is a subgroup of order 2^n (where n is the number of vertices of the simplex), of a group $oPa3o \dots 3o4o$

Rectate Where $aPoP/2oPa$ is connected to two consecutive nodes of a rectate, as $o3..a3a..3o$, then it is a subgroup of order $n + 1$, of a group where oPa is connected to $o3..3a3...3o$. There is one additional '3' in the second group.

Placing a drop of paint on one mirror will carry the image, such that each mirror connected by an odd branch will have the drop of paint on it. Each of the several different colours of mirrors, constitutes severally and alone, a separate symmetry. The size of these various subgroups, can be found from the order of the original symmetry, divided by the order of the nodes representing the selected different colours.

In the group $o3o4o3o$, the first two and last two nodes, represent separate groups, the order of which is $1/6$ or $1/2$, where $o3o4o3o$ has an order of 1152 .

8.1 The transport of Number

The number system is transferred across a mirror by reflection in its image. Thus if it falls on a ruler, it will fall on the second, fourth mirror too. By the odd numbers, it ends up on every mirror.

This creates a 'brough' number system that is the composition of all of the branches, the even branches counting as $n/2$. So the brough system of $\{3,4\}$ is $Z3Z2$, which is the ordinary integer-system. The system derived from $\{3,5\}$ is $Z3Z5$, but $Z2$ and $Z3$ are subsets of everything else, so it reduces to pentagonal nodes.

Even branches create nodes held at an incommensurable value. This means that it is not possible to superimpose equal-edged lattices or figure from nodes on both sides of an even branch. For example, the cube of unit edge has integer coordinates. But polyhedra using nodes on the opposite side of the even branch will for integer coordinates, use $\sqrt{2}$ in the edge, or vice versa.

The rate of incommensurability is the **bridge constant** of the branch, and is such that its square belongs to the number-system of the branch. The bridge-constant is co-square with $\sqrt{2+a}$ for even polygons. For numbers in $8n+6$, the bridge-constant is co-square with $\sqrt{4n+3}$, for $8n+2$, it is co-square with some ugly value, eg for 10 it's $G = \sqrt{2\frac{1}{2} + \frac{1}{2}\sqrt{5}}$.

The loop constant is the cumulation of bridge constants as one goes around a loop. It becomes part of the basic system. One use is to find out what sort of number system is used in a polygon. The even branches are used here, the branch associated with $zp21$ is found from $e42o$, which can be found from $e6o$ and $e14o$. The brough number system is $Z3Z7$, and the bridge crossings are for o , $\sqrt{3}$ and $\sqrt{7}$. At the even node, we find $\sqrt{21}$, arising from crossing a '6' bridge and a '14' bridge. In the integer system $Z[1, r21]$, the values of $\sqrt{3}$ and $\sqrt{7}$ are commensurate and the underlying system becomes $Z21$ is $Z7[1, r21]$.

In hyperbolic groups, such as $o3o4o3o6^*a$, the brough-system is found from the group $Z3Z4Z6$, which is $Z3$. Where the first two nodes are held at '1', the third and fourth nodes are held at $\sqrt{2}$. Returning to node 'a' via the last branch, we find the first two nodes are held at $\sqrt{6}$ and the last at $\sqrt{3}$. This $Z[1, \sqrt{6}]$ is the most common class-2 integer system not derived directly from a polygon.

9 The Polytope as Vector

The notion that the reflective group represents a kaleidoscope is usually read as looking through the glass end of the tube, and seeing the different patterns.

It can also be presented as an oblique coordinate system, the coordinates representing the perpendicular to some mirror. In this way, the notional coordinates of a cube $\pm 1, \pm 1, \pm 1$ would in this case give rise to one of the polytopes of this symmetry. Particularly, the axes are represented by polytopes of one marked node.

Just as x, y, z represents as a rectangular prism in the group $2,2$, the same sort of coordinate can represent a polytope of variable edges. None the same, the polytope is still *mirror-edge*, as if its vertices were carried by some kind of change-of-sign rule.

Where there is a branch greater than '2', marking either of the connected nodes will cause a polygon to appear in the hedrix of both. It carries across to the space created by any number of nodes, as long as there is a chain connecting the nodes. So while (1,0,0) is not a solid polytope in the group 2, 2, it is in 3, 5 or 3, 4. In the group 3, 2, the point (1,0,0) produces a triangle in the x-y plane, but nothing exists to lift it off that plane.

9.1 Stott Addition

Since Mrs Stott's 'expansion' operators amount to varying an axis by some amount, say 0 to 1, and vectors reflect this sort of notion, it is only fair to label the polytope-vectors here as *Stott Vectors*, and the result of additions by her name too.

In essence, the point (x,y,z) represents a *position vector*, which is actually the vector from (0,0,0) to (x,y,z). Likewise, we can suppose that the same point represents a *position polytope*.

The uniform polyhedra can then be represented by the seven non-zero coordinates, plus. Mrs Stott suggested that the snub form could be derived by alternating the coordinates of some (x,y,z), such that the figure is equilateral.

| | Tetra | Octa | Icosa | StottA | Notes |
|-------|-----------|-----------|-------------|--------|------------------|
| 0,0,1 | Tetra | Octa | Icosa | I | |
| 1,0,0 | (Tetra) | Cube | Dodeca | D | |
| 0,1,0 | (Octa) | Cubocta | Icosadodeca | ID | |
| 1,0,1 | (CubOcta) | rh CO | rh ID | I+D | rh = rhombo- |
| 0,1,1 | tr Tetra | tr Octa | tr Icosa | I+ID | tr = truncated |
| 1,1,0 | (ditto) | tr Cube | tr Dodeca | D+ID | |
| 1,1,1 | (tr Oct) | tr CO | tr ID | I+D+ID | |
| s,s,s | (Icosa) | snub Cube | snub Dodeca | | Even coords only |

Table 9: The Position vectors for each Polytope

9.2 Matrix-Dot

The stott vectors are not on an orthogonal basis, and so it is harder to derive the vector-normals for these. One can at first, convert the vectors to a right-angle system, and take the dot-product there. But that seems too much trouble, and a better solution was to be found in the matrix-dot.

In essence, we suppose $\mathbf{v} = \sum_{i=1}^n a_i \mathbf{s}_i$. The way to do dot products with this vector is to populate a matrix with $S_{ij} = v_i \cdot v_j$. For the spherical group, this turns out easy, since the vector corresponds to a ray from the centre to a vertex of a single-marked node. With a suitable supply of vectors, one can make the matrix.

In the horric or euclidian case, the vectors are all parallel, and so the matrix is simply the product of lengths. Better still, we can eliminate the dot matrix, and use a dot-product of the symmetry-vector and the target vector.

The hyperbolic case amounted to spotting the matrix, by calculating the implied vertex-figure of where two nodes are marked, eg $x_5o_3x_4o$ gives $S_{1,3}$.

The matrices were normalised to allow one to quickly write them out, without further calculations. The matrix is symmetric, and except for the bottom right corner, the column $S_{i,j} = iS(1,j)$, $i < j$.

The bottom right-hand corner is called the *animal*¹⁰, is 1×1 for 3,3,n [n integer], its value corresponds to the dimension number.

The animal for k_{11} and k_{21} are shown to the right of the same table.

The matrix is prepared by writing the vector in column 1, from bottom to the top. The next term is used for the divisor at the front. The animal is placed in the bottom right-hand column, in 3d, it may overflow the vector, it has priority over the vector.

¹⁰An animal in heraldry is a device on the field of a shield or quarter

Such a matrix forms the input screen to a Lotus 123 spreadsheet¹¹, which can calculate the length of any vector entered as an input field.

A further feature of the spreadsheet, was the ability to calculate the height of a lace prism. This was on the behest of Dr Klitzing, who had just written a paper on the segmentotopes¹². Dr Klitzing rewrote the spreadsheet to agree with the conventions of maps that he was used to. The matrix input is now auto-sizing. Such spreadsheet has found use with the members of the Polytope Discord group.

Note: This matrix corresponds to the Schläfli Matrix, but that name has been reallocated to a different process, based of the Schläfli symbol.

¹¹The actual program it was released in is in a little-known but fragile spreadsheet program called Excel, by the same people who brought you Microsoft Edlin.

¹²A segmentotope is a polytope where the vertices fall into two layers, separated by a height enabling unit edges. The pre-existing notation was to use description || description. The example of note is cube || icosahedron, which amounts to the same height as $\frac{1}{\phi}$, that is, the cube of unit edge is in an icosahedron of edge $1/\phi$.