The mathematics behind polytope theory

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Abstract

A derivation of polytope theory beginning with number theory and bases. With the derivation comes the logic behind the

1 Introduction

The study of polytopes begins with base arithmetic. There are a number of rather unexpected connections between bases and polytopes, principally arising from the cyclotomic numbers. The notation advances with some rather interesting views.

The mathematics is not itself complex, but rather it is the unstated assumptions that sets this from conventional. Somewhere, religious beliefs creep into the process.

Mathematicians design their art as to provide a quick path to the matter in question, using ordinary words for this. The question is not so much to create a unified naming, but more to lay duck-boards to the matter in hand. Often the beauty of the situation is lost.

1.1 Angles

Angles are measured as a fraction of space, and not the subtended area of the sphere. The angle from a solid section (such as at right-angles to an edge), are the same as the solid angle of the full space. So the hedric angle of an octagon is 3/8, is the same as the hedric section of a vertical edge of an octagonal prism.

1.2 Naming Conventions

I find that the names used in this subject lead to confusion and a good deal of wasted time. If as much care had been taken to giving names as to the mathematics, this problem might not arise. So in the style of Oliver Heaviside, I shall use a *rational* system of names, and set the text accordingly.

Names given in honour of people, serve to confuse and mislead later researchers. The so-called "Wythoff Notation" is an example, which might lead people to seek out Wythoff's papers, or suggest that it's "Wythoff's Notation". Both of these things I have done in front of one of the authors of that notation.

Another source of confusion, is the use of different people's names to designate part of the same process. Stott's vectors are by common names, to be normalised using Gramm matrices, and Dynkin's symbols rendered into Schläfli Matrices, even though some of these people have more important roles here.

Schläfli is the first of many to describe regular polytopes by their sillage, that is, a series of numbers representing how many around an N-2 element. There are useful processes that come from these, discussed later, but none involve the so-called Schläfli matrix. That is actually a representation of the unmarked Coxeter-Dynkin symbol in matrix-form.

In the rational notation, we shall extend a single name across an entire process, so one is aware that this goes with that.

When the dimensionality of something is increased, one must consider if the new element is a '+' or an '=' role. Where a name is taken to preserve a constant number of '=' elements, it means the term is held relative to solid. For example, *surface* is taken to represent a single $x = 1$ style operator. This is a partition of space, and corresponds to a dimensionality of N-1.

Common usage is to preserve the number of '=' signs in changes of name, while the mathematicians preserve the number of '+' signs. A face in three dimensions is '++=', that is two dimensions dividing space, but in four dimensions, it is used to represent '++==', two dimensions, while a new term *facet* is used to describe '=', eg '+++=', being three dimensions dividing space. In four dimensions, 'a tesseract has eight facets, each of which has six faces', is by no means absurd, when facet simply means a small face. The rational way is to say 'A tesseract has eight faces, each having six margins'.

It may not be such a problem, until you recall that other words have to be re-purposed or invented to fill the gap. Norman Johnson was using 'cell' in the sense of a 3d surtope, (choron). When I suggested what would he do with the word in 'cellular', he suggested 'cellule'. Such was the effects at play that this is now the 'standard term' for what people call 'cell'.

Equally offensive is 'weight' to designate the force of weight, when it correctly designates what is called 'mass' (measure). The acts specify that weighing shall take place on a balance, so the ruling equation is that of a torque balance, viz, $mgl = MGL$. The weighing is correct when both pans swing, or weigh. Given the size of the balance is such that the error in $g - G$ is significantly less than other errors, we can suppose this. The act specifies $L = l$ in the primary instance, so the weights are identical, be it in London or Darwin, or even the moon. The balance does not work in free space, which is why one is weightless. Spring scales, which correctly measure the force of weight, have to be adjusted to indicate the true weight.

Yet this does not stop large numbers of vandalisations of the correct usage on places like the Wikipedia, where students have been told by their professors that weight is a force, and so forth. A new word is needed, and *heft* is the rational choice.

2 Bases

By the enthusiast, a base represents a replacement for the current decimal system. The current state of this art is the modern decimal implementation: one has a digit for each column, and the arithmetic is implemented by tables. It is this second element which limits the choice of base. In essence, one might have to learn the arithmetic tables for b^2 separate elements.

Bases lead to measurement systems, the general rule is to clone the decimal metric system, fixing up perceived errors. Attaching a measurement system brings into play much larger numbers. While the ordinary count might bring 100 or so to mind, measurements of length range from millimetres to kilometres, often something like nine places of decimal.

For the count, a number is grouped into a number of batches of size b, and a remainder. This continues until all one is left with is remainders. Each count of remainder has its own symbol: a digit. Older number systems might have symbols to represent the order of count, repeated for as many times as the remainder. Thus 12 might become XII, meaning a batch of 10, and two remaining.

The method of converting between bases, is to replicate this casting of groups, the division is a repetition of the base, the outcome appears in the remainders. Thus, to convert 1000 to dozenal, one notes it is 83 dozen, and 4 remainder. The count of dozens is a number, and groups to 6 dozens remainder 11. Then 6 groups to nothing and a remainder of 6. The number is then 6.11.4. A symbol for the additional digits V for decimal 10, and E for 11, allows us to write this as a fairly ordinary number 6E4.

There is little mathematics in this activity. It is more a linguistic enterprise, the issue of the day is to name the extra digits, and what names the columns ought have. The example of 6E4 might be 'six hundred and elfty-four', or 'six gross, eleven dozen and four' or any of a range of issues. Part of the question is is the new base to be by itself, or is it going to live beside decimal.

Bases are taught in middle-grades, such as to children at the age of ten or eleven. It is a short session. In Pendlebury's *Shilling Arithmetic*, it occupies a half of a page. On the other hand, the decimal fractions occupy ten or twelve pages of text.

Decimal fractions are derived by successive multiplication of the fraction, the integer being the remainder, and the new fraction ensures. Unlike the count, this process usually has no end, instead, one stops when sufficient digits are found. Thus an english foot, rendered into metres goes 10 feet make 3 metres and a bit. 10 bits makes no metres and a bit. 10 bits make 4 metres and another bit. 10 bits make eight metres.

In the work of Stevin's 'La Disma'. this fraction would be 3 primes, 4 thirds and 8 fourths. The decimal or unit point would take another forty years after this work. Converting the fraction is to simply replicate this sequence of remainders, but with 12 as the multiplier.

2.1 Periods

One of the common activities that base enthusiasts like to do is to prepare a table of reciprocals of small numbers in several different bases, as if such might attract new users. Certain numbers have a terminating reciprocal, that is, they divide some power of the base. The great majority have a continuing fraction. For example, in decimal, $1/8 = 0.125$ while $1/9 = 0.1111111....$ Some reciprocals enter a period after a leader, so $1/6 = 0.1666666$, the '1' bit is a leader, and the '6' bit is the same as in two-thirds, $\frac{1}{6} = \frac{2}{3} - \frac{1}{2}$.

Since from the right of any point in the period, the digits represent some numerator on the same denominator (for example, in $1/7 = 0.142857...$, the digits beginning at the '5' represents 4/5), the pattern recurs at some number less than the denominator.

Given that $ax = b \pmod b$) and that the periods of every numerator is the same length for primes, the period must divide $p-1$. This is Fermat's little theorem. Gauss extended this by a test as to whether there are an even or odd number of loops. This is quadratic reciprocality.

2.2 Algebraic Roots

The algebraic roots are the real factors of $b^n - 1$. These exist for each divisor of n. One can find these from the decimal values, using a BIGNUM routine, the following works for roots as many as 163 digits.

The odd numbers have fewer divisors, so we follow the Cunningham Project order, of placing the odd numbers one step before the corresponding double.

o	e	decimal	+5's	equation
1		9	564	$x-1$
	2	11	566	$x+1$
	4	101	5656	x^2+1
3		111	5666	x^2+x+1
	6	91	5646	x^2-x+1
	8	10001	565556	x^4+1
5		11111	566666	$x^4+x^3+x^2+x+1$
	10	9091	564646	$x^4-x^3+x^2-x+1$
	12	9901	565456	
7		1111111	56666666	$x^6+x^5+x^4+x^3+x^2+x+1$
	14	909091	5646464	$x^6-x^5+x^4-x^3+x^2-x+1$
	16	100000001	5655555556	x^8+1
9		1001001	56556556	x^6+x^3+1
	18	999001	56554556	x^6-x^3+1
	20	99009901	5654565456	$x^8-x^6+x^4-x^2+1$
	24	99990001	5655545556	x^8-x^4+1
15		90090991	5645646546	$x^8-x^7+x^5-x^4+x^3-x+1$
	30	109889011	5665444566	$x^8+x^7-x^5-x^4-x^3+x+1$

Table 1: Small Algebraic Roots

Adding a string of 5's turns the decimal number into a form with negative digits in the base. The polynomial is then derived by replacing the digit in 10^x by $d-5$. At 163 digits, there are a few that require ± 2 as a polynomial coefficient, this process correctly handles this.

Primes that have a period of n will divide the entry corresponding to n in this table. This entry is denoted as bAn. The list of primes that divide bAn forms the 'Yates' table[1]. The balance of numbers

[1] After the corresponding table in Yates' book *Repunits*.

not dividing these are the *repeaters* and *sevenites*. Where p divides some bAj, then it also divides some $bApj$. For example, 3 divides 10A1, and so it divides 10A3 and 10A9.

A much rarer occurrence is where p^2 divides some bAn. These are the sevenites, which for any given base is extremely rare. The name refers to the smallest compound sevenite, where $7^3 \mid 18A3$. Decimal has three known sevenites, being $3,\ 487\,56598313$ run as far as 120^4. It is extremely rare to find a large compound sevenite (ie $p^3|bAj,\ p > b$), only one is known under $b < 2,000,000$ (b=68, p=113).

2.3 The base as $b^n - a^n$

2.4 Alternating Bases

It would be remiss not to discuss the historical number systems. These are not cosy numbers near ten, but rather large numbers, multiples and fractions of twenty are the order of the day, but other systems appear sparingly. Many of these systems do not use a single value per column, but alternate. The same logic applies as before, except now one must care which number produces the remainder.

Bases 18 and 20 are large enough to use a two-row abacus, and one alternates between numbers that make these. A score is made of tallies, each of units. A tally of scores might make a block. We should show the count of 118 in decimal, in both of these systems.

In base 18, a tally is 6, and a score is 18. The 118 coins make 19 tallies of 6, and four left. 19 is then bundled into 6 scores and 3. A group of 6 scores make a block. We might write this as ’ ’4, meaning a tally of scores, a tally and four.

In base 20, a tally is 5, and a score is 20. The 118 coins make 22 tallies and three, the 22 tallies give 5 scores and 2, and the 5 makes a tally of scores. We get ’ "3.

Conversion can be done directly, since each uses a series of columns. ’ "3 divides by six to give ’"4 remainder 4. Dividing this lot gives ’1 remainder ’, and ’1 gives a block of 6: ’ remainder 0. So ’ "3 (b20) is ’ ’4 (b18).

The alternating systems derive in part from having two rows on the count-table, with different widths, A count in the lower row overflows to the upper row, and the upper row overflows to the next column.

2.5 The Base as Integer System

The set `Bn` describes numbers that can be written exactly in base n. While most of the primes still remain primes, the erstwhile primes that divide n become units of the system. For example, in the set `B10`, the numbers 2 and 5 have terminating reciprocals, and thus $\frac{1}{2}$ and $\frac{1}{5}$ can be written exactly in this system.

Base enthusiasts make a good deal of the versatility of these numbers.

If one supposes that the fractional part of a number is in base $\frac{1}{b}$, then there is a direct correspondence of the numbers between 0 and 10, and the integers, by the simple ruse of reversing the index. 12345, for example, corresponds to 5.4321. This closely represents the correct way to calculate the size of the fractional part, since it often meets the line of all a/N is represented for a = 1 to N, at powers of any size.

This means that while any binary number can be exactly represented in decimal, this does not directly confer to rulers. The decimal 1/1000 contains all of the eighths, but the binary ruler of this order would go to 1/1024. The order of intersection is thus calculated as a *cascade*. Decimal numbers contain binary numbers, at the same rate that they contain numbers made of binary digits (0, 1), but allow for ten digits. So it corresponds to $\ln 2/\ln 10 = 0.30103$. A ruler divided into 1000 parts, contains $1000^{0.30103}$ parts, or 8 in total.

Where the bases contains several primes, the proper cascade is found by adding the least portions for each. For example, with 120 and 10, the primes are 2, 3, 5, in 120 we find that there is actually 8 for 2, so we get

$$J = \min(\frac{\ln 8}{\ln 120}, \frac{\ln 2}{\ln 10}) + \min(\frac{\ln 3}{\ln 120}, \frac{\ln 1}{\ln 10}) + \min(\frac{\ln 5}{\ln 120}, \frac{\ln 5}{\ln 10})$$

$$= \min(0.43438, 0.30103) + \min(0.229475, 0) + \min(0.3361756, 69897) = 0.63720555$$

This produces an intersection of 81.583 for 1000 in decimal, or 21.1278 for 120, which is different to the cumulation of 2-5 numbers. This is because the decimal digits is smaller than the twelfty, and more digits are needed to show the same large N.

3 Chord or Iso-Arithmetic

The element in chordal arithmetic, is to take a step, and veer $2a$ to the right, and then take another step of the same size. The bisector to the right is at angles b to the first step, and this in general produces a series of isosceles triangles of angles $2a$, $b\,b$. The ray connecting the beginning of the first step to the end of the second makes an angle to each step equal to a.

The points reached by this method form a circle. Such is evident that the radius of said circle are formed by the equal edges of the isosceles triangle. The product of any two chords gives a number which can be expressed as a sum of chords. The set over 'any sum of' is the **span** of that set, and it suffices to show closure of multiplication to make the set one of an *algebraic integers*.

A given chord is made from steps at bearings starting at $(n-1)a$, and proceeding in steps reducing the angle by $-2a$, until $a.(1-n)$ is reached. Thus a chord of three steps is made at angles $2a$, 0, and $-2a$. The product of two chords comes from replacing each step of one chord, with a series of steps representing the other chord. The product of such chords produces a rectangular table, with the rows running from $(n-1\ +m-1)a$ across to $(n-1\ +\ 1-m)a$, and the last row running from $(1-n\ +\ m-1)a$ to $(1-n\ +\ 1-m)a$.

The run of chords to make the sum, is for the longest chord to take the first row, and down the last column, and by successive reductions of 2, each new chord is the next row, down the next column, until the last chord taken is part of the last row. So the product of two chords, of m and n, is a sum of chords running from $m+n-1$ to $m-n+1$, where $m>n$. This is the sum of n chords centred on chord of m steps.

The full set of chords can be derived from C2, which is a, $-a$. Multiplying Cn by C2 produces the sum of two chords $\mathtt{C(n+1)}+\mathtt{C(n-1}$

Since all chords for a given polygon can be derived from C2 alone, we shall refer to C2 as the **shortchord**. In convex polygons, it is the shortest chord not actually an edge. It is denoted by a lower-case letter, usually a, but where several polygons are in hand, P has shortchord p, and so forth.

The chordal triangle runs as powers of a, but is given in tabular form, rather than as algebraic expressions. It runs thus. The expressions that polygons like that of 7 or 11 sides, can be solved by the process of equating two consecutive chords that add to these numbers.

The table is by additions, the number in cell (i,j) is given by (i-1,j-1) - (i-2,j). For example, the four in the C7 row comes from the last '1' in the C6 row, less the -3 in the C5 row.

The second table shows the powers of a in terms of the chords.

						0	0 C0 vertex	g	f	e	d	c	b	a	1	(0
						1	1 C1 edge							1	0	(1)
					1	0	a C2 shortchord						1	0	1	(2)
				1	0	-1	b C3					1	0	2	0	(3)
			1	0	-2	0	c C4				1	0	3	0	2	(4)
		1	0	-3	0	1	d C5			1	0	4	0	5	0	(5)
	1	0	-4	0	3	0	e C6		1	0	5	0	9	0	5	(6)
1	0	-5	0	6	0	-1	f C7	1	0	6	0	14	0	14	0	(7)

The equations that odd polygons must satisfy can be found by equating consecutive chords of the table.

zp{5} set $C2=C4$ gives $x^2-x-1=0$

zp{7} set $C3=C4$ gives $x^3-x^2-2x+1=0$

zp{11} set $C5=C6$ gives $x^5-x^4-4x^3+3x^2+x-1=0$

zp{13} set $C7=C6$ gives $x^6-x^5-5x^4+4x^3+4x^2-x-1=0$

The snub polyhedra can be catered for by imagining they be $3,\mathtt{S_p}$. In this case, the rake of triangles form a series of chords. The shortchord of $\mathtt{S_p}$ comes by supposing the third or fourth chord is the non-triangular figure. In the antiprisms, this leads to a quadratic, but for the snub dodecahedron and snub cube, a cubic ensures.

Antiprisms: `C3` $= a(p) \qquad x^2 - 1 - a = 0$

Snub figures: `C4` $= a(p) \qquad x^3 - 2x - a = 0$

3.1 Sample Numeric Series

The progression of chords resembles a number of famous numerical series. Before we look at isoseries in general, we shall look at some famous series.

3.1.1 J3 - Fibonacci-like Series

The fibonacci series runs 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ... Each number is the sum of the two to the left, so $5 = 2 + 3$. One can construct fibonacci series from any pair of starting numbers, the iteration is $t_{n+1} = tn + tn - 1$. It will be noted that the power series is $x^2 = x + 1$ is the convergence of terms, this is $\phi = \frac{1}{2} + \frac{1}{2}\sqrt{5}$

That the series above starts with $F_0 = 0, \ F_1 = 1$, can be found from that every F_{mn} is a multiple of F_m, only when these settings are held. Because of this, it resembles a series of rep-units, such as $1, \ 11, \ 111, \ \cdots$. This is less remarkable, when one supposes its resemblance to the chords of polygons, the multiples feature due to inscribed polygons.

The series representing the shortchords of inscribed polygons, is the Lucas series. It runs as 2, 1, 3, 4, 7, 11, 18, 29, 47, ... Each of these numbers has a square that differs exactly four from a fifth-square, for example $11^2 = 5 \times 5^2 - 4, \ 18^2 = 5 \times 8^2 + 4$. In practice, $L_n^n - 5 \times F_n^2 = \pm 4$.

The alternating members of this series more closely resembles a polygon of shortchord of three.

2	3	7	18	47	123	322	843	$y^2 = 5x^2 - 4$
1	4	11	29	76	199	521	1364	$y^2 = 5x^2 + 4$

The top row are numbers whose squares are 4 greater than a fifth square, the lower represents numbers whose square is four less than a fifth square. In both cases, we see that $t_{n-1} + t_{n+1} = 3t_n$, which is consistent with a polygon of short-chord 3. Steps of two on either row show that there is an inscribed polygon obeying $t_{n-2} + t_{n+2} = 7t_n$, and so forth.

3.1.2 J4 - The Heron series and the Mersenne primes

The Heron formula for the area of a triangle, given its three sides, is $A = \sqrt{s(s-a)(s-b)(s-c)}$, where $2s = a + b + c$. The Heron series relates to a series of triangles, with three consecutive numbers for edges, which give an integer area. This is $A = \sqrt{3n.(n+1).n.(n-1}$ or $n\sqrt{3(n^2-1)}$, where the edges are $2n-1, \ 2n, \ 2n+1$. Thus we seek instances where $n^2 - 4 = 3a^2$.

The series runs 2, 4, 14, 52, 194, 724, 2702, ... The values are twice the middle row here. The iteration for this series is as follows:

1	0	1	4	15	56	209	780	2911
$\sqrt{3}$	1	2	7	26	97	362	1351	1364
$1 + \sqrt{3}$	1	3	11	41	155	571	2131	7953

This series is also used to find large primes, reckoned as the Mersenne test. Suppose that $4n + 1$ is a number, which divides the heron-number at position n. If the power-of-two divisor of $4n$ is larger than the non-power of 2, that is $4n = k.2^x, \ k < 2^x$, then the number $4n - 1$ is a prime. The further proviso is that $4n - 1 = 7 \pmod{21}$. It is most famously implemented when $k = 1$ in the above example, but this is an extreme requirement.

The primes that work run as *7*, *31*, 79, 103, *127*, 151, 223, 271, ... The algorithm used to determine the members of the heron-series, is typically finds members for 2^n only.

The algorithm works because primes $p = 7 \pmod{14}$ are 'upper long' in J4, and the algorithm creates the quarter-period, which is for all isoseries, zero. So if the period is zero at n, it means that the number is a product of numbers of the form $24a + 7$ or $48a + 1$, but the power of two prevents any factorisation. Hence the number is prime.

Let's look at the proof that 103 is prime. First we have $103 = 7 \pmod{24}$, which means that it's an 'upper long'. This means its period divides $p + 1$ (ie upper), an odd number of times (ie long). Because we are playing with the equivalents of $2\cos\theta$, the period of 103 divides 104 an odd number of times, and so at 26 and 76, (the right-angle points), the cosine is 0, and so it should divide this.

If we calculate the series to 26 terms, we get heron(26)= 742 397 047 217 294. This is a multiple of heron(2), representing quarter-points of 13 cycles of 8, but it is also a multiple of 103.

Now, in order to have a period that is 8 times an odd number, we find by gaussian quadratic rule, that $p = 1,\ 7 \pmod{48}$. So 7, 97, 103 are the first three primes, and we know 7 does not go into 103, and anything past this is more than the square root, then 103 is prime.

In fact, 742 397 047 217 294 is 2*7*103*103*4998431569. The last of these is a lower-short prime, its period (104) divides p-1 (lower) some 48061842 times (which is even).

The presence of 103 twice in the product is rare. No other prime under 120^4 has this property. It is similar to Shank's prime (487) in decimal, which divides its own period.

3.1.3 J5 - Notes on the rational use of 2

3.2 Isoseries

Isoseries can begin at any value, since the algorithm is $t(n+1) = a \times t(n) + t(n-1)$. This is symmetric, so if $S(n)$ is a series, then so is the reverse $S(-n)$. The series can be shifted any number of places, so if $S(n+m)$ is also a series. It admits scalar multiplication and addition, so given two series starting at $(0,1)$ and $(1,0)$ it then the series $(a,b) = b(0,1) + a(1,0)$ at every place.

Less obvious is that steps of m in the series, forms an isoseries in a new constant, and one particular series represents the isopowers, that is, $p(n)$ is the constant at n steps. It is also represented as $p\hat{}\hat{}n$.

$$\begin{array}{cccc} i & ii & iii & iv \\ t(n+m-1) & = a(m-1)t(n) & -t(n-m+1) & a\hat{}\hat{}m-1 \\ t(n+m) & = a(m)t(n) & -t(n-m) & a\hat{}\hat{}m \\ t(n+m+1) & = a(m+1)t(n) & -t(n-m-1) & a\hat{}\hat{}m+1 \end{array}$$

Since columns i and iii represent isoseries in a, so must column ii, which is the sum. Dividing through by $t(n)$, the constant at each step is an isoseries of a. It requires little further to puzzle out that $a(0) = 2$ and $a(1) = a$. The result behaves like a power-series.

3.3 Isoseries in 10.1

It is instructive to construct isoseries in this number, because it shows quite clearly what can be had from this operation.

P/0	2.0	c0	0.0
P/1	10.1	c1	1.0
P/2	100.01	c2	10.1
P/3	1000.001	c3	101.01
P/4	10000.0001	c4	1010.101
P/5	100000.00001	c5	10101.0101
P/6	1000000.000001	c6	101010.10101

Table 2: Isopowers and Isochords base 10.1

When the base 10 is set to a value where $x^p = -1$, the number 10 then represents $\mathrm{cis}(1/2p)$ in rational units, or $\mathrm{cis}(\pi/p)$ in natural units. Powers of 10^n then are steps of $\mathrm{cis}(n/2p)$. The deflection of a step of 1 from the direction of 100, is an angle of $1/2p$, which is the margin-angle of a polygon {p}.

The Isopower series gives rise to the shortchord[2] for successive inscribed stellations of the polygon, some of which may be inscribed polygons. For this reason, we can represent the power as {P/n}, representing that it is an inscribed polygon of {P}.

The isochord gives the succession of n steps, rotating by $1/2p$ at each step. Such describes the chords of a polygon, in the given form, all such chords would fall on the real axis. The thing to note here is that these values look like repunits of a base, or number constructed entirely of 1's. They are indeed

[2]The shortchord is the base of a triangle, formed by two edges of a polygon.

repunits of base 100, expressed in base 10. It will be noted that steps of n in the chord table, will give an inscribed polygon {P/n} of edge $c(n)$.

Another series to be considered is the pseudo-isoseries, which takes the form of $t(n+1) = k(n) + t(n-1)$. The fibonacci and lucas are examples, with $k = 1$. Various forms with $k = 2$ give rise to the silver ratio $(\sqrt{2}+1)$. These can be constructed as a pair of interlaced isoseries.

P/0	2	2		C0	0	0	
P/1	10		10	C1	1		1
P/2	102	102		C2	10	10	
P/3	1030		1030	C3	101		101
P/4	10402	102*102-2		C4	1020	102*10 - 0	
P/5	105050		102*1030-10	C5	10301		102*101-1
P/6	1060902	102*10402-102		C6	104030	102*1020-10	

Table 3: Pseudo Isopowers and Isochords base 10

3.4 The Isoladder

The isoladder is an algorithm which allows one to find arbitrarily large members of an isoseries, in logrithmic time. It runs at $\frac{2}{3}$ of the speed that is used to find ordinary large powers.

In the following example, it is desired to find the 37th member of a series beginning with terms t0, t1 using a shortchord s1.

i0	i1	s2	t0	t1	t2	t3
37	1	1	0	1	2	3
18	0	2	1	3	5	-
9	1	4	1	5	9	13
4	0	8	5	13	21	-
2	0	16	5	12	37	-
1	1	32	5	37		

Table 4: Calculating $s2\hat{}\hat{}i1$ from a series beginning $t_0,\ t_1$

The values i1 and i2 are integers, the registers needed for these need not be large. The variables s2, t0 - t3, are done in a bignum process. This could mean doing a modulus step at each calculation.

The algorithm runs along these lines. This is REXX[3] code that runs this. For odd i0, t0 and t1 end up with t1 and t3. For even i0, t0 and t1 become t0 and t2. The % is integer division, the // is integer modulus.

The procedure works with exact integers, rather than approximation, and thus a version of the algorithm can be made to use modular arithmetic. The additional argument `b1` is the modulus of the calculations. So while Isoquad can work with real numbers, isomod is designed for BIGNUM type calculations.

ADDQUAD is for series of the form $t_{n+1} = kt_n + t_{n-1}$. Given t_0, t_1 and n, the input is `t(n)=ADDQUAD(k, n, t(0),t(1`
For generalised fibonacci numbers, k=1.

ISOQUAD is the series $t_{n+1} = kt_n - t_{n-1}$. Alternating members of an addquad series make an isoquad series, which is the reason for the `return isoquad()` call at the end.

```
addquad: procedure; parse arg g0, i0, t0, t1
if i0 < 1 then do; i0 = 1-i0; t2 = t1; t1 = t0; t0 = t2; g0 = -g0 ; end
i1 = trunc(i0/2); i2=i0-i1-i1
if i2=0 then t2=t1 ;
 else do; t2 = mmv(g0,t1,-t0); t0 = t1 ; end
t1 = mmv(g0,t2,-t0); g0 = mmv(g0,g0,-2); i0=i1
return isoquad(g0, i0, t0, t1)
```

[3] ReginaREXX is available for most platforms. The code is SAA REXX

```
isoquad: procedure ; parse arg g0, i0, t0, t1
if i0 < 1 then do; i0 = 1-i0; t2 = t1; t1 = t0; t0 = t2; end
do forever ; if i0=1 then leave
i1 = trunc(i0/2); i2=i0-i1-i1
if i2=0 then t2=t1 ;
 else do; t2 = mmv(g0,t1,t0); t0 = t1 ; end
t1 = mmv(g0,t2,t0); g0 = mmv(g0,g0,2); i0=i1
end /* forever */
return t1

mmv: procedure; parse arg x0, x1, x2 ; return x0*x1-x2
```

When the modulo relative to some number P (which can be as large as $P * P <$ NumericDigits) is needed, simply copy these routines, then rename the above two functions addquad and isoquad to something different, and call mmp() for mmv()

```
mmp: procedure expose p; parse arg x0, x1, x2;
r1 = x0 * x1-x2; r2 = trunc(r1/p)
r2 = r1 - r2 * p; if r2 < 0 then r2 = r2+p
return r2
```

Here are some sample routines that access isoquad or addquad.

```
fibon: procedure; parse arg t0; return addquad(1,t0,0,1)
lucas: procedure; parse arg t0; return addquad(1,t0,2,1)
qdiag: procedure; parse arg t0; return addquad(2,t0,1,1)
qdiag: procedure; parse arg t0; return addquad(2,t0,1,1)
heron: procedure; parse arg t0; return isoquad(4,t0,2,4)
ipow:  procedure; parse arg t0, t1; return isoquad(t0, t1, 2, t0)
icho:  procedure; parse arg t0, t1; return isoquad(t0, t1, 0,  1)
```

fibon(n) returns the nth fibonacci number, starting 0,1,1,2,3,5,8...
lucas(n) returns the nth lucas number, starting 2,1,3,4,7,11,29
qside(n) returns the approximate square side 0,1,2,5,12,29,70,169
qdiag(n) returns the corresponding diagonal 1,1,3,7,17,99,239

Note that each of these four convert from a pseudosequence to alternating isoseries at the initial call.
heron(n) returns the middle edge of a triangle x-1, x, x+1, of integer area. It runs 2,4,14,52,194.
ipower(b,n) and ichord(b,n) return the nth term of the isopower and isochord series for any base.
imod(b,n,m) returns isopower for ipowb,n modulo m. The Messerine prime test can be demonstrated with this command, as

```
  mesprime:; procedure; parse arg p;
  if imod(4,2**(p-2),2**p-1)=0 then say p 'is prime'; return
```

3.5 Isobases

The isobases are derived from isoseries in integers, essentially add to the variety of base available. Since we have already shown that these produce rep-unit numbers, and additional outside powers, we simply construct the root from $a^2 + b^2 = k$, and $ab = f$. Instead of requiring a and b to be integers, we suppose that k and f are integers, and describe the base as k/f.

One of the advantages of this approach is there is no loss in generality in writing ordinary base 10 as 101/10, or 3/2 as 13/6. The same program can be used for both regular bases and isobases.

We can then modify the algebraic roots to work from an isobase of 10.1.

The Algebraic root for 30 is 5665444566, in symmetric form. This is 55555555555 + 109889011. Because we are working from both ends, it is possible to remove leading fives (which count as zero), and fold the number in two.

```
halfstr:  procedure; parse arg str ;
str1 = trim(str,'5')                    /* str1 = "665444566" */
strx = length(str) %% 2 + 1             /* strx = 5           */
parse var str1 1 strf (strx) .          /* strf = "66544"     */
parse var str1 . (strx-1) strb .        /* strb = "44566"     */
if strf = reverse(strb) then return strf; else return Error
```

The equality of the two halves is demonstration that the digits have not strayed too far from ± 0. If there had been a carry, it would have affected to the left on one side, and to the right in the reverse.

Having found that 44566 is the required string, the second part is to evaluate the number. The columns that go through to row 6 are isopowers. These are added to the running total at the rate of $d-5$, where d is the extracted value. The first two give $d-5=-1$, account for the negative sign. The third is 5, leaving the number unchanged, the last two are 6, giving $6-5=1$, adds the digit in.

0	1.0	4	-1.0	1	-1
1	10.1	4	-11.1	10	-11
2	100.01	5	-11.1	98	-11
3	1000.001	6	988.901	970	959
4	10000.0001	6	10988.9011	9602	10561
5	100000.00001		.	95050	
6	1000000.000001		.	940898	

Table 5: Isoroot and Algebraic root for 30

If we don't want decimals here (and simply use BIGNUM integers), then we should use isopowers of 101, and multiply the cumulative sum by 10 before adding the next digit (including 0). The actual values for 101 and 10 are k and f in the base itself.

Having pulled a number at the end of this process, it remains to set a command to invoke factorisation. This can be handled by `'factor' x`, or where `factor=c:\bin\factorx.exe`, simply `factor x`. REXX is pretty good at formating the output and program inputs.

3.6 Factorisation Notes

Having written such a program, it is a simple matter to create a list of these root strings and the matching root, eg `30: 44566`, and simply let the program have its head. Prowling through the output, it is easy to make certain observations.

- The factors of the numbers are usually primes for which r divides $p+1$ (upper) or $p-1$ (lower). Small primes divide this root if $p \mid (r)$ means $p \mid (pr)$ and vice versa. This can be seen in decimal, where $3|111$ although the period if 3 is one digit, not three digits.
- The division of the period into the maximum, is governed also by a gaussian rule. That is, for fibonacci numbers, $17|17+1$ an even number of times, as 17 divides F_n when 9 divides n.
- Isobases are much more accessable, because the roots are smaller.

The gaussian rule is a form of quadratic reciprocal, $g(p,n)$ is even when $p \mid x^2 - n$ for some x and odd if it doesn't. It makes no sense to evaluate (p,p) When even and odd are represented by ± 1 the numbers are weakly multiplive, that is $g(pq,n) = g(p,n)g(q,n)$. It is only necessary to find the values for the co-squares, since $g(p,n) = g(p,nx^2)$.

The primes are positive or negative as they are 1 or 3, modulo 4. The cases for 1 and 2 are given in both forms (1 is square). The co-square factor must agree in sign, so for 120, the co-square is 30, and the product is -2*-3*+5, one uses those rows.

The cycles for -5 and -6 are against modulo 20 and 24, demonstrate the property that if the number is relatively prime to 10 and 12, and the tens-digit is odd, then it has an odd number of odd-tens numbers, and an even tens-digit gives rise to an even number of tens-divisors.

For bases. a $g(b,p)=1$ means that p has a short period, ie divides (p-1) an even number of times. Where $g(b,p)=-1$, the period length divides p-1 an odd number of times.

p	$g(p,n) = 1$	$g(p,n) = -1$
-1	1, 5 (mod 8) (1 mod 4)	3, 7 (mod 8) 3(mod 8)
+2	1, 7 (mod 8)	3, 5 (mod 8)
-2	1, 3 (mod 8)	5, 7 (mod 8)
-3	1	2
+5	1,4	2, 3
-7	1, 2, 4	3, 5, 6
-11	1, 3, 4, 5, 9	2, 6, 7, 8, 10
+13	1, 3, 4, 9, 10, 12	2, 5, 6, 7, 8, 11
+17	1, 2, 4, 8, 9, 13, 15, 16	3, 5, 6, 7, 10, 11, 12, 14
-19	1, 4, 5, 6, 7, 9, 11, 16, 17	2, 3, 8, 10, 12, 13, 14, 15, 18

Table 6: Gaussian function g(p,n) for all x mod p

For isobases, when $g(b^2-4,p) = 1$, the prime is lower, ie its period divides $p-1$. When $g(b^2-4,p) = -1$, the prime is upper, meaning its period divides p+1. When $g(b+2,p) = 1$, the dividend is even, when $g(b+2,p) = -1$, the dividend is odd.

Negative numbers, in both cases are taken to account.

The fibonacci numbers form the rep-units of an isobase -3. We then have upper/lower according to -1*-5 = +5, and long/short according to -3+2=-1. This means that in every odd position, the fibonacci number is a product of (2) and numbers of the form $4x+1$. Numbers ending in decimal 1, 9 still divide $p-1$, while those ending $3, 7$ are long. So 13, 17 are lower-long, so their periods divide 7 and 9. We find the seventh fibonacci number is 13, and the nineth is 34 = 2*17. For 11 and 19, these are lower long, so their periods divide 10, 18 an odd number of times. fibon(10) = 55, and fibon(18) is 2584, which is a multiple of fibon(9) = 34, by Lucas(9) = 76.

The messerine primes rely on that $2^p - 1$ is an upper long prime in isobase 4. short/long is decided by -3 * -1, and upper / lower by -3 * -2. The relevant primes have a modulus of 7, mod 24, and so it is upper (-1*-1) against 3 and long against 6.

3.7 Isoroots

These are the isoroots, the list produced by a feed into a symbolic calculator (Derive for DOS). The shortchords for the polygons solve the equation for $2p$ being set to 0.

```
 6: X - 1
 4: X
 3: X + 1
10: X^2 - X - 1
12: X^2 - 3
 8: X^2 - 2
 5: X^2 + X - 1
14: X^3 - X^2 - 2*X + 1
18: X^3 - 3*X - 1
 9: X^3 - 3*X + 1
 7: X^3 + X^2 - 2*X - 1
15: X^4 - X^3 - 4*X^2 + 4*X + 1
20: X^4 - 5*X^2 + 5
24: X^4 - 4*X^2 + 1
16: X^4 - 4*X^2 + 2
30: X^4 + X^3 - 4*X^2 - 4*X + 1
22: X^5 - X^4 - 4*X^3 + 3*X^2 + 3*X - 1
11: X^5 + X^4 - 4*X^3 - 3*X^2 + 3*X + 1
21: X^6 - X^5 - 6*X^4 + 6*X^3 + 8*X^2 - 8*X + 1
26: X^6 - X^5 - 5*X^4 + 4*X^3 + 6*X^2 - 3*X - 1
28: X^6 - 7*X^4 + 14*X^2 - 7
36: X^6 - 6*X^4 + 9*X^2 - 3
```

```
42: X^6 + X^5 - 6*X^4 - 6*X^3 + 8*X^2 + 8*X + 1
13: X^6 + X^5 - 5*X^4 - 4*X^3 + 6*X^2 + 3*X - 1
```

3.8 The Pell equation $X^2 = kY^2 + 4$

The Pell equation in the form $X^2 = kY^2 + 4$ gives rise to an isoseries, which at most is an inscribed series to the minimal solution. The matter is to find the minimal X for a given k. In turn, this involves finding a continued fraction for $\sqrt{k}$, which can be done without any knowledge of it.

```
Let base = 21; and k = sqrt(21) = 4.58257569496

4.58257569496 = 4 + 1/1.71651513898 = 4 + ( -4 + k)           2
1.71651513898 = 1 + 1/1.39564392376 = 1 + ( -1 + k) / 5       2
1.39564392376 = 1 + 1/2.52752523152 = 1 + ( -3 + k) / 4       4 = 1 *  2 +  2
2.52752523152 = 2 + 1/1.89564392421 = 2 + ( -3 + k) / 3      10 = 2 *  4 +  2
1.89564392421 = 1 + 1/1.11651513840 = 1 + ( -1 + k) / 4      14 = 1 * 10 +  4
1.11651513840 = 1 + 1/8.58257573850 = 1 + ( -4 + k) / 5      24 = 1 * 14 + 10
8.58257573850 = 4 + x               = 8 + ( -4 + k)         110 = 4 * 24 + 14

Where 110 = the cube of 5, also, 110 * 110 - 24 * 24 * 21 = 4.
```

On paper, the calculation runs as follows. `bb` is the supplied constant, and `gx` is the truncate of the square. The sought value is returned in `b3`.

	a	b	c	e	f	g
0		2				
1		0	0		1	
2	4	2	4	5	5	
3	1	2	1	20	4	
4	1	4	3	12	3	
5	2	10	3	12	4	
6	1	14	1	20	5	
7	1	24	4	5	1	*
8	4	110				

```
gx = trunc(sqrt(bb))
bo = 2; b1 = 0; c1 = 0; f1 = 1
  do forever
  a2 = (gx + c1) % f1
  b2 = a2 * b1 + bo
  c2 = a2 * f1 - c1
  e2 = bb - c2 * c2
  f2 = e2 % f1
  if f2 = 1 then leave
  bo = b1; b1 = b2;
  c1 = c2; f1 = f2; end
b3 = b2 * c2 + b1
```

This algorithm needs further checking. The first is that $b3 * b3 > b2 * b2 * bb$, something that can occur if the number is comprised of primes of the form $4n + 1$. In this case, we would we would put $b3 = b3 * b3 + 2$.

The second fix requires that the algorithm works for even $b3$, and we need to check that this is actually something of the form $b3 = b4^3 - 3 * b4$. This is an isocube. In effect, we take the cube root of $b3$, truncate it, add one, and test for the value above.

Printing values out at the end of these two tests gives

bb	a	b	c
5	4	18	3
7	16	16	16
17	8	66	66
21	110	110	5

```
a  value produced per algorithm
b  after first test (b3 * b3 > b2 * b2 * bb)
c  after second test (b3 = b4 ** 3 - 3 ** b4)
```

The interesting observation here is that `c` never exceeds the square root of `bb`. This places a limit on how many iterations the loop can pass through (ie twice the square root).

A test was put in to see if there is no remainder on dividing `e2%f1`. No error has arisen in the many thousands of values enumerated in this algorithm.

4 Spans of sets and equations

A span of a set S is defined as any member $\sum z_i s_i$, where z_i is an integer and s_i is a member of the set S, If every $s_i s_j$ is in the span of S, then the span constitutes an integer system. The proof lies in the existence of a trimex $S_{i,j,k}$, whose columns $S_{i,j} = s_i s_j$. The determinate of the matrix $S_{i,j,k} v_i$ is an integer, the factors of which, represents the divisor to the 1/r power, where r is the free rank.

It follows that the intersection of a finite-span closed to multiplication, and the rationals, is the integers.

The set defined for a span of an equation, is a (real) root, and its powers. Where the set is governed by $x^3 = 2$, the span is of $1,\ x,\ x^2$. The transition from one root to another is an internal isomorphism, when $x' \in \operatorname{span} S$ and external if it isn't.

The span of the shortchord of a polygon p is such, that which solves the isoroot for $2p$. The various stellations provide the alternate roots, and so every polygon forms an integer system that is isomorphic within itself.

It will be noted that the final digit is one, unless the polygon is $2p^n$, for which it is p. From this one finds that it is never possible to reach the centre of a polygon p^n, by taking unit steps of any combination of rational angle.

The product of chords of a circle gives $\prod_1^{n-1} \operatorname{cho}(a/n) = n$. Since every chord comes from an integer system as described above, the integer-factor of the chord is 1 for composite numbers, and $1/(p^{x-1}(p-1))$ for a power of a prime. Any other integer-factor implies the angle is irrational.

4.1 The Pentagon

The pentagon answers to the equation $x^2 - x - 1 = 0$, which we can implement as a stone-table, where coins in two adjacent columns, become one in the column to the left, viz $11 = 100$ (in the same base).

4.2 The Hypercomplex plane

This is graphic representation of a system where $j^2 = 1$, in much the same style as the complex plane. The various class-2 systems, like the pentagonal, octagonal and dodecagonal systems fall on it.

It acquires the hyper- prefix, by virtue that trig functions on the ordinary complex plane become hyperbolic trig functions here. All of the functions are replicated, except now things like $\cos + i \sin$ become $\cosh + j \sinh$, and circles of given radii become replaced with hyperbola of given differences.

There are unit hyperbola, and a corresponding -1 hyperbola. The units of any system lie on these hyperbolae. A rotation in the plane becomes a hyper-rotation, that is an area-preserving skewing of the shape along the hyperbola.

Also present are the zero and alt-zero axies, which represent the parts of the 0-hyperbola. The perpendicular distance from these two axies, represent the real and alt-real size of things. An octagon in the real space becomes by inversion, an alt-octagon.

Numbers then have two signs, one real, and one alt-real. The distances observed in finite polytopes have squares whose sign and alt-sign are both positive. The ones with +- signs have an piecewise finite[4], and a sparse[5] Further, because there is an isomorph, by replacing j with $-j$, the points are restricted to a finite area where both the real and alt-real squares fall in a definite ranges.

4.3 The Heptagon

5 The Radiant Solid

A solid is taken to be a region of space, occupied by the substance of the figure, for which there is a definite boundary, which within it, it is present, and without it, it is absent. One can consider obvious non-solid distributions, such as the Gaussian dot, answering to $d = \exp(-\operatorname{rss}(x, y, z))$

[4]Piecewise-finite means that it is possible to construct the incident surtopes on any given surtope. An example is $\{\frac{5}{2}, 4\}$.

[5]Sparse here means that every point of space is no more than some finite x from a vertex, and no two vertices are closer than y from each other.

The prototype solid is taken to be a double-unit sphere[6] which answers to the equation $r^2 = x^2 + y^2 + z^2$, or $r = \text{rss}(x, y, z)$. Here `rss` stands for root-sum-square. Such a figure has a clear surface at 1, values less than this are interior, those greater are exterior.

The solid is taken as a function of surface. Specifically, where a ray crosses the surface a number of times, each instance is found separately. So, the sphere is not taken as a position of angle $\text{fn}(r, \theta)$ but of surface, that is, $\text{fn}(r, S)$. The integrals are over surface, not over angle.

The surface is taken as $n = \nabla d$, that is, the gradient of the density vector. Because the solid has no motion, $\oint \mathrm{d}S$.

The enclosed volume is the moment of surface, that is $v = \oint \nabla x \cdot \mathrm{d}S$. Because the volume is independent of where the moment is measured from, it follows that $\oint \mathbf{n} \cdot \mathrm{d}\mathbf{S} = 0$

If the surface were to be broken into two parts, the ring[7] separating the two parts span a definite vector area, and that is independent of the closing surface. The sum of the two halves must always be 0.

The radiant function, then for each value of x, defines a copy of the solid at the scale x, where 1 is full-sized. When x is set to $\frac{1}{2}$, then the result is a half-sized figure.

5.1 The radiant space

For radiant space is one where the point $(x,\ y,\ z)$, represents a copy of the cartesian product of the solids in these spaces, at the radiant function. the solids we draw in this space is formed by the integration of points. Each of these axies can represent separate solids of any dimension.

The generic prism product is represented by the cube with the diagonal (0,0,0) to (1,1,1). The radiant function is $\max(x, y, z)$, where the maximum is over the absolute value. This corresponds to cutting out of the X, Y and Z spaces, the shape so represented. The word *prisma* is as off-cut.

A cylinder can be represented as a stack of coins, each coin is replicating the bottom base at that height. It can also be represented as a faggot of matches, where each match represents the full height and a point on the base.

The radiant function $\text{sum}(x,\ y,\ z)$ represents a different product. The product applied over unit edges gives the orthotope, with the canonical vertices $(\pm 1, 0, 0)$. This product covers the surfaces of the various base figures, in much the same way that a tent covers its pegs. Originally, the word for tent was put forward, but *tabernacle* is already taken. Instead, a word for *cover* was chosen, this being assimilated to **tegum**.

Unlike the prism product, the tegum is a *drawn* product, or one of *draught.* The allusion here is to as gum might draw into strands as the two parts are separated. The interior does not take part.

The pyramid product was found from the surface of the tegum. It corresponds to the plane $X+Y+Z = 1$, which draws the surtopes at the corners into a progression of prisms, the size of the prisms being unity.

5.2 Coherent Products

In a measurement system, the coherent units of space are the prism product of the length measure. Where different products used, different coherent units may arise.

For the space described above, one can start with a prism or tegum, and replace any axis with a different shape of the same dimension. Since this will cause each layer of the product to increase proportionally to the old volume to the new volume, the volume of the product is the product of the volume. To this end, the sphere, cube and octahedron has been so tested, and lead to separate coherent volumes. Note that the volumes are not equal, the octahedral line is only 1/6 of the cubic line.

It is then possible to write a new physics, using these measures, especially the tegmic units, as base units. The measurement of volume from area becomes $V = \int \mathbf{r} \cdot \mathrm{d}\mathbf{S}$, and one might need to reevaluate the relation of inverse-square laws. For example, the rational form of the coulomb equation becomes $F = \frac{cQQ}{8\pi r^2}$, the extra factor of 2 comes from the surface of a sphere is now $S = 8\pi r^2$.

---

[6] In real life, the circles and spheres are measured by their diameter. Radius is only used for small arcs of a circle, such as deviations and range.

[7] A ring is a closed boundary on the surface.

	Pn	Cn/Pn	Cn	Cn/Tn	Tn	Cn/Tn
1	linear	1	diametric	1	diagonal	1
2	square	$4/\pi$	circular	2	rhombic	$2/\pi$
3	cubic	$6/\pi$	spheric	6	octahedral	$1/\pi$
4	biquadric	$32/\pi^2$	glomic	24	tetrtegmic	$3/4\pi^2$
n	Prismic	$\eta^{n/2}/n!!$	Crind	$n!$	Tegmic	$\eta^{n//2}/(n-1)!!$

Table 7: The Coherent Products

5.3 Altitude

Altitude refers to those axies over which draught and copy happen, like that of the antiprism or pyramids. An antiprism is the drawing of a figure onto a parallel copy of its dual.

If one supposes a three-dimensional altitude, with antiprisms forming the axies, then one will note that the cover is a tegum product. None the same, any face of the octahedron-in-altitude is dual to its opposite face. and thus the tegum-product of antiprisms is itself an antiprism.

Likewise, the dual case, of the prism-product of antitegums (dual of antiprisms, but has its own construction), is itself an antitegum. The construction of an antitegum is to project cones over parallel dual figures, and take the intersection. But in the prism, the opposite corners of the cube-in-altitude represent the pyramid product of the figures, the opposite is the pyramid product of the dual, and hence the intersection is an antitegum (for each great diagonal of the cube).

Using large dimensions as the altitude lead to very large dimensions. A lacing of lines in this manner can lead to a five-dimensional solid quite easily. The table below shows a triplet of rectangles, parallel in one axis and orthogonal in the other, set at the corners of a triangle ABC. The result is five-dimensional, with just 12 vertices.

A	x	x	o	square
B	x	o	x	square
C	o	x	x	square

The pyramid product might be understood in this light, but where the bases appear only once along the main diagonal.

5.4 Surface, Periform, Hull

The boundaries of a solid vary when the surface is let cross itself. The example here refers to the small stellated dodecahedron $\{\frac{5}{2}, 5\}$.

The **hull** is the least convex shape that contains the points in question. The vertices of the stellated dodecahedron is an icosahedron.

The **periform** is the shape equal to the referenced points of a solid, excluding all exterior points. It is the shape that you would make in modeling a polytope. The periform is an apiculated dodecahedron, with pyramids raised on each face.

The **surface** is the gradient of density, and its vector-moment is the volume of the polytope. There can be parts of the surface that are interior to the periform, such as the pentagons in the pentagrams, which are a d2 (density-2) wall separating endocells of d1 and d3.

5.5 Endocells

A surface that crosses itself, will divide the interior into a number of different cells, each cell has its own density.

6 The Schläfli Series

If a is the short-chord of the polygon, the diameter D of the same polygon can be found from the relation $D^2 = \frac{4}{4-a^2}$. The maths is simplified by working with the squares, rather than the actual lengths. This happens if you have no way of finding the square root[8].

[8] In some parts of the world, calculators in the 1970s were still expensive things.

In this calculation, if P is a polygon, then p is its shortchord-square. Learning the values of P and p was a way of getting around not having the necessary trig and log tables.

Where $\{P, Q\}$ is a polyhedron, then its vertex-figure is $p\{Q\}$. It follows from this that the diameter can be found by using the same formula as above. The iteration in this table is by eg $\{P, Q, R\} = 2\{Q, R\} = p\{R\}$

$$\begin{array}{cccc} & & 2 & 2 \\ S & s & \frac{4}{4-s} & 4-s \\ RS & \frac{4r}{4-s} & \frac{4-s}{4-r-s} & 8-2r-2s \\ QRS & \frac{q(4-s)}{4-r-s} & \frac{4(4-r-s)}{16-4q-4r-4s+qs} & 16-4q-4r-4s+qs \\ PQRS & & 32-8p-8q-84-8s+2pr+2ps+2qs & \end{array}$$

This function is symmetric, in that f(p,q,r) = f(r,q,p). From this function alone, one can calculate the radius of the polytope as $f(verf)/f(figure)$. It turns out that the function of a product is the product of the function, so $f(a)f(b) = f(a \times b)$. The final column of the table gives the Schläfli function for `3,3,..P`. The value `A` is for $\mathtt{k}_{1,1}$, while `B` equates to $\mathtt{k}_{21}$.

When this index first comes to zero, from positive values, the result is a euclidean tiling. We might note that in 3D, the condition for $16 - 4q - 4r - 4s + qs = 0$ can be rewritten as $4r = (4 - q)(4 - s)$, it is the factor 4 on the right-hand side that severely limits the values that lead to a polyhedron with rational angles. We should bring to bear some powerful theory to finish that proof.

Polygon	P	a	a^2	Sch {3,..,P}
2	R	0	0	$2n$
5/2	V	0.61803398875	0.38196601125	
3	S	1	1	$n+1$
4	Q	1.41421356238	2	2
$\mathtt{k}_{1,1}$	A		2	4
$\mathtt{k}_{2,1}$	B	1.5	2.25	$9-n$
5	F	1.61803398875	2.61803398875	$2-(n-1)/\phi$
6	H	1.73205080757	3	$3-n$
7		1.801937736	3.2469796037	
8			3.41421356237	
10		1.90211303259	3.61803398875	
12		1.93185165259	3.73205080757	
	U	2	4	

Table 8: Commonly used shortchords of polygons

The values for 3..3,5 and 3..3,6 can be written as $\phi^2 - n/\phi$ and $3 - n$, where Ⓡe initial value represents the square of the shortchord.

6.1 Antiprisms

Antiprisms answer to the third chord being the base, that is, $x^2 - 1 - p = 0$.

	a	$4a^2$	Remark
C2	1	1	Tetrahedron
2.5		1.61803398875	
C3	1.41421356268	2	Octahedron
C4		2.41421356238	
C5	1.61803398875	2.61803398875	Icosahedron
C6		2.73205080757	
U	1.73205080757	3.00000000000	Triangle stripe

Table 9: The indicated shortchords for antiprisms as {3,P}

It is interesting to note here that the shortchord-squares include values that have a complex conjugate, that is, the usual rules of isomorphism do not apply here.

6.2 Snub polyhedra

The snub cube and snub dodecahedron can be realised as polyhedra of the type {3,sC} and {3,sD}. Using the circumaadius of the figures, and the formula $a^2 = 3 - 1/(-1 + r^2)$, where r is the circumradius, we get these indicated shortchords.

p	circumradius	a	a^2
3,2	0.707106781186	1.41421356238	2.00000000000
sT	0.951056516295	1.61803398875	2.61803398875
sC	1.343713373744601	1.685018324889720	2.839286755,214160
sD	2.155837375115564	1.715561499697367	2.943151259,243881
3,6	very large	1.73205080757	3.000000000

Table 10: The indicated shortchords for snubs as {3,P}

6.3 The cube

The Schläfli equation for the polychora {p,q,r} is given by $pr - 4p - 4q - 4r + rs$. Setting this to 0, for Euclidean lattices, gives $4q = (4 - p)(4 - r)$. Since all of these represent polygons, the product on the right must be a multiple of 4. But we can demonstrate that the only solutions that work is when $p = r = 2$, viz {4,3,4}, or when $pr = 0$, ie {2,q,r}.

The next highest power of 2 in polygon shortchords is 8, where the effective power of 2 in the shortchord is $\sqrt{2}$.

6.4 The system `B2Z4`

This family is suggested by the octagons and octagrams of the uniform octahedral group. In essence, one extends these figures by supposing that only some faces are accessible, and that, for example, the octagons of the truncated cube belong also to some of the faces of a regular solid with octagon faces.

Likewise, the squares of the rhombocuboctahedron might be replaced by octagons, that a polyhedron with a girth of eight octagons arise.

6.5 The Schläfli Function

Schläfli put forward a request or hope, that one day, the order of a group might be derived from the symbol. Sixty years later, Coxeter[9] reported no further enlightenment. It still has not been attended to. Instead, the group order is found by way of the Euler characteristic for polytopes of that group, or where the polytope occurs as a face or cell of a tiling.

Note that if the polytope is treated as a surface, varying between elliptic and spheric, for example, such a function is less likely to be found as a matter of course.

It is known from spherical excesses and defects, the form of such equation, but experience shows that it is irrational in the odd dimensions of surface.

7 Eutactic Lattices

Coxeter defines a *eutactic star* as the normals to the mirrors, in either direction, of the mirrors of a group. Groups with only odd mirror-angles or right angles, have only one star. Each separate set of nodes that removal of even branches leaves, is a separate star.

My scheme lists the kaleidoscope, rather than the group it follows. A symmetry identified by several letters means that there are several distinct mirrors, each subset of these also constitute a group.

[9] Regular Polytopes, 1947

Name	Cox.	Curr.	Kri.	Order	Lattice	Cox.	Cur.	Kri.
Rectangular			r	2^n	prismatic			rr
Simplex	A	A	s	$(n+1)!$		P	A	t
Halfcube	B	D	h	$2^{n-1}n!$	qtr.cubic	Q	D	q
Cubic	C	B=C	hr	$2^n n!$	semi.cubic	S	C	qr
					cubic	R	B	qrr
Polygonal	D	I	p#	$2p$	horogon	W	I	rr
Hexagon		G	ss	12	hexagonal	V	G	tt
Gosset	E	E	g	gos(n)	gosset	T	E	y
	F	F	hh	1152	3,4,3,3	U	F	qq
Pentagonal	G	H	f	pen(n)				v

gos(n) is the product of the first n of 1,2,6,10,16,27,56,240.
pen(n) is the product of the first n of 2,5,12,120

Table 11: Reflection-groups in all dimensions

A group like `4qq` contains 24 mirrors, twelve of each 'colour'. A single mirror of the second `q` reflected through the first `q` will only be imaged to four copies. This would produce `qr`. The mirrors of `q` are made from three copies of `r`, one needs mirrors from at least two sets of these. The group {3,4,3} is by itself, made from two mirrors from each `q`, these being the first two nodes, and then the third and fourth node.

7.1 Eutactic Stars

The eutactic lattice is thence the span of the eutactic star. The interest here is that every Wythoffian mirror-edge polytope is contained in its relative lattice. These lattices have as sections, eutactic lattices of lesser dimension, and for as far as nine dimensions, may be constructed as layers of balls. This represents the twin problem of efficient sphere-packing and the *kissing number*, or equal spheres touching a common sphere.

From these structures come lace towers of different polytopes, and also the form of efficient non-lattice packings.

The *stations* of the lattices are where all of the mirrors cross. This happens at more points than the lattice may occupy, and as such represent 'fractional coordinates'.

The lattice occupies one of these positions, but in a stack of layers, the lattice can be placed at different standing points [10]. From such layers, we can find all sorts of exciting things.

The number of standing points or stations, is equal to the number of separate nodes that have the maximum symmetry. For `t`, the group is represented by a polygon or loop of branches, with $n+1$ nodes. For `q`, the group is represented by a chain of branches, with two sub-chains, of unit length, branching of the second and second last nodes, or 4 in total. For `y`, the groups are 2_{22}, 3_{31}, and 5_{21}, these have 3, 2 and 1 maximal node, or $9-n$ nodes.

We shall find that if s represents the number of stations, the packing of spheres of diameter $\sqrt{2}$ is $1/\sqrt{s}$ for these lattices.

7.2 The `t` lattice

This is formed by a lattice over the edges of a simplex. It's essentially an oblique cubic: if one vertex is taken as the origin, and the rest as units in each dimension, the full set of points is the integer coordinates.

The volume of a simplex, in tegum units, can be found from a cube corner. This simplex, bounded by the origin and the units of axis, (ie 1,0,...0 etc), forms a volume of one unit tegum measure. This is the same as the base by height, if the simplex is taken as the base, and the height is the perpendicular to the origin. The distance is given by the ray to the coordinate $(\frac{1}{n}\frac{1}{n}\cdots)$ or $\frac{1}{\sqrt{n}}$. A simplex of edge $\sqrt{2}$ has a volume of $\sqrt{n}$.

The volume of a simplex in n dimensions is thus $1/\sqrt{1+n}$, since the above calculation if for a simplex-face of an orthoplex or tegum-power.

[10] hence 'station'

We now suppose there are rays from the origin to the mid-points of the surtope of the opposite face. The product of these rays, one of each kind, give the volume of the simplex in tegum units. The unit of the lattice, is however, the rhombus formed by these rays. These are extended to the points (1,1,1,..,0,0), where the number of 1's represent the number of vertices of the simplex surtope. These range from 1 to n, giving a product of rays of $n!\sqrt{n+1}$ in tegum-units, or $\sqrt{n+1}$ for prism-units.

The walls between the cells in this rhombotope, is given by planes that sum to integers ($\sum x_i = m+1$, and correspond to m-rectate of the simplex. The simplex is itself taken to be the 0-rectate.

The stations of this lattice fall at the centres of the cells. This divides the long ray of the rhombotope into n sections, and hence $n+1$ stations, counting the ends.

7.3 The Laminate Tilings of types LB2, LPA2 and LPB2

In two dimensions, the lattice is that of triangles. The three stations fall at the vertices, the centres of the 'up' triangles, and of the 'down' triangles. If we imagine the lattice is made of spheres, the next layer can fall into say, the up holes. The down holes become up holes, and the vertices become down holes.

The tetrahedra of the oct-tet truss so formed, lie between the vertices and the down triangles, and between up holes and the vertices of the new layer. Octahedra form between the up triangles and the down triangles.

Progressing in this way, the lattice for 3t is made. This is also LA2, as the vertices advance around stations of 2t.

If the layers rock between two stations, the packing formed is 'hexagonal close-pack', forming a different uniform lattice LB2. This means the layers switch backwards on each layer, rather than advancing through

Two more uniform tilings arise if between each layer of LA2 and LB2, a prismatic layer is inserted. The effect is to place the next layer of balls directly on top of the previous one. Such would give rise to LPA2 and LPB2 arrays.

A vertex, for falling on a side of a lamina, can only admit two kinds of laminae. and as such, the layers that have the eutactic for t is a layer of the next dimension, or a prism layer of such.

The remaining laminate tilings in three dimensions, are LC2 and LPC2. These are constructed on a square-lattice etching[11]. The layers have a triangle section, and basically bunt or shift the layer of squares a half-cell in one direction. By using internal vectors in the cells, it is possible to have a uniform tiling that cycles the axies instance by instance.

The alternate layers of triangles and squares is LPC1, the triangles move the horogon layer back and forwards, is C1, the P is the layer of squares, which preserve the layers.

7.4 The semicubic lattice q

Although the cubic lattice is the iconic form of lattice, packing spheres in a cubic form is very inefficient. Removing alternating spheres halves the number of spheres in the space, but allowing the remaining spheres to expand in diameter from 1 to $\sqrt{2}$, means that from three dimensions onwards, the packing is $\sqrt{2^{n-2}}$ times more efficient than the cubic.

In general, this lattice has four *stations*, or standing points where the same mirrors serve the symmetry. These points correspond to the centres of the cross polytope opposite the semicubic, and the two semicubics of the body-centred positions. These are arranged in order as alternating between the cubics and the semicubics of the dual. Thus if 0123 represent the four stations of the q them 02 would be one cubic and 13 would be the other. Note these positions are relative. If you decide to occupy position 1, then the four stations become 3012.

The semicubic is reckoned as a packing of spheres of diameter $\sqrt{8}$. The opposite is at a distance of $\sqrt{4}$. The other two are in the halfcubes of $\sqrt{n}$. In three dimensions, the halfcube is a tetrahedron, which is smaller than the octahedron. But in higher dimensions (over four), it is the halfcube that forms the largest cell, in eight dimensions, this is deep enough to take a full semicubic itself. The resulting lattice here is the gosset-lattice E8.

[11] An etching is the cell walls that meet the surface of the layer, such that in place of a blank plane, one with cells is presented to the next layer.

8 An Overwiew of Notations

Coxeter's book *Regular Polytopes* does not describe polytopes as vectors. Instead, the notation is simply the Schläfli symbol, with serves until the discussion on the Gosset polytopes. In order to make these text, one supposes that the Schläfli symbol is a representation of the Dynkin diagram, and that the branches of these groups derive from a curtail, this: $\{{}^{3\ 3}_{\ 3}\}$. Such figure is further written as 0_{21}. The various Gosset polytopes are then written with a succession of leading '3's, eg $\{3,3,{}^{3\ 3}_{\ 3}\}$ for 2_{21}. The Elte figures can start from any end, such as $1_{2,2}$, with the exception of 1_{42}, which fails Elte's rather artificial definition.

The Schläfli symbol has no notion of anything further than what is written. While it might correspond to the branches of a Coxeter-Dynkin diagram, the Schläfli symbol has no notion of anything on nodes. It is for this reason that the Stott operator was included.

One can linearise these groups, by supposing the shortest branch begins not with a '3', but a node "3, which would mean to count backwards two nodes. Alternately, one could use letters A and B to connect the new node to the second or third last node. The polytope 2_{21} becomes `4B`. Using a number to represent a string of '3's, then means that something like {3,3,5} could not be so expressed. the fix for this was to denote these by letters too: `Q` for '4', and `F` for '5'. So {3,3,5} becomes `2F`.

The new names for the polytopes ought not contain superscript or subscripts, or brackets, or quotes. These are meant so that one can use the name as a subscript $R_{0,name}$ or set the clear limit of the name, as 'name' that the reader might know what is and isn't the name. Brackets are meant, as in mathematics, to denote the enclosed is a single object, thus in $(3+a)*b$, the bracketed expression is reduced to something that can be multiplied onwards. Ideally, subscripts and superscripts are best avoided completely.

The final allocations of symbols was based on being able to describe the second extension[12]. Exactly where Wythoff's construction fits in is not known, but it's based on a few scattered comments in 'Regular Complex Polytopes'. These were also to be included.

The symbols are designated as *structural* and *decorative*. Structural elements build the kaleidoscope, two diagrams with the same structural items are the same symmetry. Decorative elements create an object for the kaleidoscope to reflect.

8.1 "Wythoff" Notation

This has no connection to Anton Wythoff. It is instead, an 'honour-name', the main purpose appears to mislead and distract researchers. In essence, it's a decorated Schwarz-triangle, with the mirrors bisecting edges appearing before a vertical bar '|'. It is used in Mangus Wenninger's *Polyhedra Models*.

8.2 Stott-Schläfli Notation

The more common notation, and one that works in higher dimensions, is to use the modified Stott expand notation against a regular polytope denoted by a Schläfli symbol. The modification to Stott's system is to start off with a zero-size regular solid, rather than a size-1 one. The regular solid is then made by expanding surtope-0, or pushing the vertices radially outwards. The raw Schläfli symbol serves as a name for the regular polytope.

Where Mrs Stott wrote e_1C_{600}, the new form becomes $t_{0,1}\{3,3,5\}$. Mrs Stott's notation already supposes an expanded form of node 0, where the revised notation does not.

An alternate SS notation is to suppose that the individual expands run as powers of 2, from 1, 2, 4, 8..., and that the figure in question is denoted by a dimensional letter and polytope base. We have A, B, C, D... representing the dimensions from 1 upwards, and then t, o, c, i, d representing the polyhedra in 3D, or the equivalent polychoron in 4D. There is an extra 4D regular, which is given the letter q.

A figure such as the example above gives Di3. This is a four-dimensional ..3,5, with expansions at vertices and edges. The prism-products are simply the concatenated symbols, such as a dodecahedral prism is ACd1. The polygons are Bp, the antiprisms are Cp. The figures are then arranged into the list according to the first instance of the polytope.
Some others are duplicates. Ct4 and Ct6 are Ct1 and Ct3. Also Ct2, C5 and Ct7 are Co1, Co2, and Co3. Ct8 ia Ci1. In general, 4 and 6 are the same as 1 and 3 of the dual, so Co6 is Cc3.

In 4D, the positions 4, 8, 10, 12, 13 and 14 are the same as 2, 1, 5, 6, 11 and 7 of the dual, and referred to as such. So Dq8 is the same as Dq1. 6 and 9 and 15 are identical from either end, but are

[12]The first extension is the compact hyperbolic groups, the second is the paracompact groups.

Key	2D	3D	4D	Remarks
A	B4	Cc1	Dc1	Measure Polytopes
Bn	Bn	ABn	AABn	Polygon cube prisms
BB			BpBp	polygon-polygon prisms
C		Cp	ACp	polygon antiprisms
Cx1		Cx1	ACx1	Platonic figures
Cx2-7		Cxn	ACxn	ME archimedean
Cx8		Cx8	ACx8	Snub Cube Cc8 and Dodeca Cd8
Dx1			Dx1	Regular polychors
Dx2-15			Dxn	Mirror-edge archimedian
Dx16			Dx16	snub 24ch Dq16 and grand antiprism Di16

Table 12: Catalog of Uniform Polytopes

constructed from the base polytope, rather than the medial (6). Do2, Do5, and Do7 are Dq1, Dq2, and Dq5.

8.3 Kepler names

Kelper's names for the various archimedean polyhedra make some sense. However, they do not generalise all that well, and the words loose their meanings in some of the applied schemes. The numbers refer to the Stott-index of the previous section.

platonic (P) (1 = v) The platonic figures, by a generic face-count. These might be distinguished from other figures of the same face-count, by saying 'regular' P.

truncated P (3 = ve) The vertices and their verge is cut off the platonic figure, leading to a doubling of edges of the original faces.

snub (8) A twisted figure made of triangles, the non-triangular faces belong to that of the named polytope (Cc8 and Cd8). The same pattern is followed in four dimensions, with the Dq16 'snub 24choron'. Gosset provided this name.

middle (2 = e) A figure derived from bisecting the edges of platonic figures. They fall in pairs, so Cuboctahedron, Icosadodecahedron.

rhombo-M (5= vh) A notional intersection between a middle-figure and its dual. The dual has rhombic faces (rhombo-dodecahedron and rhombo-tricontahedron), which is the source of 'rhombo' here.

truncated-M (7=veh) Notionally a truncation of the middle-figure, except that the proper truncation would have rectangles, rather than squares. Alsp *rhombo-truncated.*

These names are fine in three dimensions, but in four dimensions and higher, things come a little undone. Little more than the truncate survives unchanged.

The antitegmal sequence is the intersection of duals, as one increases and the other decreases. Mapped on a higher dimension, these intersections represent slices of an antitegum of either end. The intersection produces an aggressive vertex bevel, which has the effect of moving the vertices along the edges, then when the edges are exhausted, towards the centres of the hedra, until exhaustion, and so forth, until all of the surface is worn away, and the vertices proceed towards the centre as those of the dual.

The process in three dimensions passes through truncated cube, cuboctahedron, truncated octahedron. The Cuboctahedron is middle of this series. In four dimensions, the process adds two extra steps. The edge centres of the tesseract do not align with those of its dual. Instead, the middle-point is half-way between.

The remaining two (rCO, tCO), correspond to a truncation of the cuboctahedron, until its edge centres are met. In practice, the cuboctahedron has a $1 : \sqrt{2}$ rectangle with no degrees of freedom, but topologically, the truncated and rhombo-Cuboctahedron serve as a third truncate.

base (1)(8) The base polytope, and its dual (8).

truncate (3) (12) The edges are shortened in situ, creating new faces at the vertices.

rectified (2) (4) The edge-centres of one are the hedron-centres of the other.

bitruncate (6) The middle form is now comprised of the truncates of the vertex-figure duals.

cantellated (5) (10) This is the rectified rectate. The bicantellate is the rectate of the birectate.

cantitruncated (7) (14) The truncate of the n-rectate gives the n-cantellate.

runcinate (9) The antiprism figure gives rise to this in 4D. Norman used the term to denote node 3, I use it to denote the last node.

runcitruncate (11) (13) This is not the truncate of the runcinate, or any other figure. It's not an easy row to hoe here, so the figure is just given some sort of 'fake' construction.

omnitruncated (15) This one has a vertex for each flag, and represents the extreme bevelling on every node. Such figures usually represent the Cayley diagram of the symmetry group. The dual is the vaniated figure.

The last four items do not easily come from Kepler-style truncations, but use operators that were first used by Mrs Stott. The fake Kepler-style names are due to Norman Johnson.

Likewise the duals can be suitably named. The following list is due to the author, but the notes describe some alternate names.

base (1) (8) The duals of regulars are also regular.

apiculate (12) (3) Such rise peaks or pyramids on the faces of the base. Where the cube stands for `1`, the apiculated cube becomes `12`. The Kepler-style names is to use the Greek for thrice to five times (trikis, tetrakis, pentakis) as the pyramid is raised over a triangle, square or pentagon.

surtegmate (2) (4) The face of these are formed when the faces of the apiculate meet in pairs on the base of the pyramid. This gives an *edge-margin* tegum, which in three dimensions is the rhombus. In three dimensions, one uses rhombo-*face count* hedron, giving dodecahedron and tricontahedron.

bi-apiculate (6) The faces here consist of pyramid products of the edge of one of the figures, and the margins of the other face.

bi-surtegmate A bi-surtegmate has bi-edge times bi-margin (ie second of each), is the dual of the bi-rectate. No examples occur in four dimensions.

strombiate (9) The faces of this are antitegums of the vertex-figure faces. The topological net of this is as if one were to product the dual's surface dividers onto the figure. Such faces would lie around the line connecting the figure's vertices to the centre of incident faces, or the vertices of the dual.

vaniated (15) The faces are the individual flags as they strike the surface. The faces of this figure represent in regular groups, the individual symmetry cells, and the margins form the various mirrors.

9 The Laws of Symmetry

While one can do some fancy mathematics about joining mirror-groups together, the necessary laws to walk to the major subgroups are as follows. All of these rules are completely reversible, so by rule 1, we can split `c3a4o3o3o` into four branches c3a branching from c, and from *o3o4a3c3o*, three branches from c in the chain c3o, each new branch c3a.

Bisection If nodes $A, B, C \cdots$ are of the same kind, and that branches $AB = BC = AC \cdots = x$, and all the branches to other nodes $d, e, f, \cdots$ are such that $Ad = Bd = Cd \cdots,\ Ae = Be = Ce \cdots,\ \cdots$, then nodes $A,\ B,\ C,\ \cdots$ are equal, and that any number of these can be replaced by a single node A, connected to a $2n$ branch, and then as many 3 branches as needed to use all of the selected nodes.

Antiprism If the structure `aPoP/2oPa` for any value of P, is connected to a chain of 3-branches $a3o\cdots 3a$, then this is a subgroup of order 2^n (where n is the number of vertices of the simplex), of a group $oPa3o\cdots 3o4o$

Rectate Where `aPoP/2oPa` is connected to two consecutive nodes of a rectate, as o3..a3a..3o, then it is a subgroup of order $n+1$, of a group where oPa is connected to o3..3a3...3o. There is one additional '3' in the second group.

Placing a drop of paint on one mirror will carry the image, such that each mirror connected by an odd branch will have the drop of paint on it. Each of the several different colours of mirrors, constitutes severally and alone, a separate symmetry. The size of these various subgroups, can be found from the order of the original symmetry, divided by the order of the nodes representing the selected different colours.

In the group o3o4o3o, the first two and last two nodes, represent separate groups, the order of which is 1/6 or 192, where o3o4o3o has an order of 1152.

9.1 The transport of Number

The number system is transferred across a mirror by refection in its image. Thus if it falls on a ruler, it wall fall on the second, fourth mirror too. By the odd numbers, it ends up on every mirror.

This creates a 'through' number system that is the composition of all of the branches, the even branches counting as n/2. So the through system of {3,4} is `Z3Z2`, which is the ordinary integer-system. The system derived from {3,5} is `Z3Z5`, but `Z2` and `Z3` are subsets of everything else, so it reduces to pentagonal nodes.

Even branches create nodes held at an incommensurable value. This means that it is not possible to superimpose equal-edged lattices or figure from nodes on both sides of an even branch. For example, the cube of unit edge has integer coordinates. But polyhedra using nodes on the opposite side of the even branch will for integer coordinates, use $\sqrt{2}$ in the edge, or vice versa.

The rate of incommensurability is the **bridge constant** of the branch, and is such that its square belongs to the number-system of the branch. The bridge-constant is co-square with $\sqrt{2+a}$ for even polygons. For numbers in $8n+6$, the bridge-constant is co-square with $\sqrt{4n+3}$, for $8n+2$, it is co-square with some ugly value, eg for `10` it's $G = \sqrt{2\frac{1}{2} + \frac{1}{2}\sqrt{5}}$.

The loop constant is the cumulation of bridge constants as one goes around a loop. It becomes part of the basic system. One use is to find out what sort of number system is used in a polygon. The even branches are used here, the branch associated with zp21 is found from e420, which can be found from e60 and e140. The through number system is `Z3Z7`, and the bridge crossings are for 0, $\sqrt{3}$ and $\sqrt{7}$. At the even node, we find $\sqrt{21}$, arising from crossing a '6' bridge and a '14' bridge. In the integer system $Z[1, r21]$, the values of $\sqrt{3}$ and $\sqrt{7}$ are commense and the underlying system becomes `Z21` is `Z7[1,r21]`.

In hyperbolic groups, such as o3o4o3o6*a, the through-system is found from the group `Z3Z4Z6`, which is `Z3`. Where the first two nodes are held at '1', the third and fourth nodes are held at $\sqrt{2}$. Returning to node 'a' via the last branch, we find the first two nodes are held at $\sqrt{6}$ and the last at $\sqrt{3}$. This $\mathtt{Z}[1, \sqrt{6}]$ is the most common class-2 integer system not derived directly from a polygon.

10 The Polytope as Vector

The notion that the reflective group represents a kaleidoscope is usually read as looking through the glass end of the tube, and seeing the different patterns.

It can also be presented as an oblique coordinate system, the coordinates representing the perpendicular to some mirror. In this way, the notional coordinates of a cube ± 1, ± 1, ± 1 would in this case give rise to one of the polytopes of this symmetry. Particularly, the axies are represent by polytopes of one marked node.

Just as x, y, z represents as a rectangular prism in the group zp2,2, the same sort of coordinate can represent a polytope of variable edges. None the same, the polytope is still *mirror-edge*, as if its vertices were carried by some kind of change-of-sign rule.

Where there is a branch greater than '2', marking either of the connected nodes will cause a polygon to appear in the hedrix of both. It carries across to the space created by any number of nodes, as long as there is a chain connecting the nodes. So while (1,0,0) is not a solid polytope in the group 2, 2, it is in 3, 5 or 3, 4. In the group 3, 2, the point (1,0,0) produces a triangle in the x-y plane, but nothing exists to lift it off that plane.

10.1 Stott Addition

Since Mrs Stott's 'expansion' operators amount to varying an axis by some amount, say 0 to 1, and vectors reflect this sort of notion, it is only fair to label the polytope-vectors here as *Stott Vectors*, and the result of additions by her name too.

In essence, the point (x,y,z) represents a *position vector*, which is actually the vector from (0,0,0) to (x,y,z). Likewise, we can suppose that the same point represents a *position polytope*.

The uniform polyhedra can then be represented by the seven non-zero coordinates, thus. Mrs Stott suggested that the snub form could be derived by alternating the coordinates of some (x,y,z), such that the figure is equilateral.

	Tetra	Octa	Icosa	StottA	Notes
0,0,1	Tetra	Octa	Icosa	I	
1,0,0	(Tetra)	Cube	Dodeca	D	
0,1,0	(Octa)	Cubocta	Icosadodeca	ID	
1,0,1	(CubOcta)	rh CO	rh ID	I+D	rh = rhombo-
0,1,1	tr Tetra	tr Octa	tr Icosa	I+ID	tr = truncated
1,1,0	(ditto)	tr Cube	tr Dodeca	D+ID	
1,1,1	(tr Oct)	tr CO	tr ID	I+D+ID	
s,s,s	(Icosa)	snub Cube	snub Dodeca		Even coords only

Table 13: The Position vectors for each Polytope

10.2 Matrix-Dot

The stott vectors are not on an orthogonal basis, and so it is harder to derive the vector-normals for these. One can at first, convert the vectors to a right-angle system, and take the dot-product there. But that seems too much trouble, and a better solution was to be found in the matrix-dot.

In essence, we suppose $\mathbf{v} = \sum_{i=1}^{n} a_i \mathbf{s}_i$. The way to do dot products with this vector is to populate a matrix with $S_{ij} = v_i \cdot v_j$. For the spherical group, this turns out easy, since the vector corresponds to a ray from the centre to a vertex of a single-marked node. With a suitable supply of vectors, one can make the matrix.

In the horric or euclidian case, the vectors are all parallel, and so the matrix is simply the product of lengths. Better still, we can eliminate the dot matrix, and use a dot-product of the symmetry-vector and the target vector.

The hyperbolic case amounted to spotting the matrix, by calculating the implied vertex-figure of where two nodes are marked, eg x5o3x4o gives $S_{1,3}$.

The matrices were normalised to allow one to quickly write them out, without further calculations. The matrix is symmetric, and except for the bottom right corner, the column $S_{i,j} = iS(1, j),\ i < j$.

3,3,3	1	2	3	4	5,6,7,8
3,3,4	q	2	2	2	2,2,2,2
3,3,5	f	2	3-f	4-2f	5-3f
3,3,A k_{11}	2	2	4	4	4,4,4
3,3,B k_{21}	3	2	4	6	5,4,3,2
3,4,3	2	4	2q	q	

A

n	n-2
n-2	n

B

2n-2	n-1	2n-6
n-1	4	n-3
2n-6	n-3	n

Table 14: Stott Matrix Vectors and Animals

The bottom right-hand corner is called the *animal*[13], is 1×1 for 3,3,n [n integer], its value corresponds to the dimension number.

The animal for k_{11} and k_{21} are shown to the right of the same table.

The matrix is prepared by writing the vector in column 1, from the bottom to the top. the next term is used for the divisor at the front. The animal is placed in the bottom right-hand column, in 3d, it may overflow the vector, it has priority over the vector.

The body of the matrix is filled out with the second, third, fourth etc multiples, as far as, and including, the diagonal. The matrix is symmetric, and this allows the rest of the matrix to be filled.

The divisor is applied after the matrix, is $\frac{2}{d}$ where d is the overflow value, if diameters are sought, or $\frac{1}{2d}$ for radii.

An example of construction for the matrix of the 4B or 2_{21} group.

	4							4	5	6	4	2	3
	5							5	10	12	8	4	6
	6							6	12	18	12	6	9
$\overline{3}$	4			10	5	6	$=\frac{2}{3}$	4	8	12	10	5	6
	2			5	4	3		2	4	6	5	4	3
	3			6	3	6		3	6	9	6	3	6

Table 15: Stott Matrix, Left, the vector and animal, Right, complete.

The vector is written in column 1, from the bottom to the top, and overflows into the numerator. The animal is written to the bottom right-hand corner, here evaluated for $n = 6$.

In the right, the empty columns below and including the diagonal are multiples of column 1. the values above the diagonal are filled as the matrix is symmetric.

Note: This matrix corresponds to the Catalan matrix for undirected groups, but I am not sure of the extent of meaning.

10.3 The Dynkin Matrix

A need arose, whereby it was desirable to calculate the result of a reflection in any plane. In essence, what vector v has a dot product of 1 with itself, and 0 with all other vectors. Such a matrix would consist of column-vectors, which when multiplied by the stott matrix, gives the identity matrix.

The experimental values showed a matrix with 2 as the diagonal, and the negative of the shortchord of the angle between the planes i and j, occupying D_{ij}.

The dynkin matrix is then comprised of vectors, for which the value $d_i \cdot d_j$ would give the cosine of the angle between them. Since these represent the normal unit to the plane, the angles between them are the supplement of the angle between the mirrors themselves.

In essence, we can use a matrix inversion to calculate the Stott matrix, and said matrix can be found by entering the negative shortchord for angles into the appropriate cells. Such matrix has a value that correctly matches the corresponding Schläfli value, is given below, for again the group 2_{21} or `4B`.

$$D_{ij} = \frac{1}{2}\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}$$

Values directly under the diagonal indicate consecutive nodes are linked by a branch whose shortchord is $D_{i,i-1}$. The value in the final column is due to the `B` branch connecting the nodes as marked: `2/2B/`, or `o3o3x3o3oBx`.

Such a matrix forms the input screen to a Lotus 123 spreadsheet[14], which can calculate the length of any vector entered as an input field.

[13] An animal in heraldry is a device on the field of a shield or quarter

[14] The actual program it was released in is in a little-known but fragile spreadsheet program called Excel, by the same people who brought you Microsoft Edlin.

A further feature of the spreadsheet, was the ability to calculate the height of a lace prism. This was on the behest of Dr Klitzing, who had just written a paper on the segmentotopes[15]. Dr Klitzing rewrote the spreadsheet to agree with the conventions of maths that he was used to. The matrix input is now auto-sizing. Such spreadsheet has found use with the members of the Polytope Discord group.

The name *Schläfli Matrix* is applied to this matrix, based on a passing remark that Coxeter made in Regular Polytopes. The matrix has nothing to do with Schläfli. Using this name brings the same word into two different processes for finding the diameters of the polytopes.

11 Rotational Groups

Hamilton[16] gave one of the earliest descriptions of the alternating group A_5 as $A^2 = B^3 = C^5 = 1,\ C = AB$. This is the snub dodecahedron, with the faces marked clockwise, and the snub triangle in reverse. The same definition, with $C^5 = 1$ replaced with $C^4 = 1$ or $C^3 = 1$ gives rise to the snub cube and snub tetrahedron (ie octahedron). These represent the closest non-involuntary groups, that is, groups where the primary elements have order greater than 2.

If you eliminate the C in this group, you get $A^2 = B^3 = (AB)^5 = 1$, for which the Cayley diagram[17] is a truncated dodecahedron, with the decagons anticlockwise, and the triangles clockwise.

12 Regular Complex Polytopes

The regular complex polytopes were described as a kind of Coxeter group, where the edges might have an order greater than two. The regular complex polygons are derived as an extension of the ordinary polyhedral groups.

The group $p[2q]r$ is defined as $A^p = B^r = (AB)^q$. Coxeter found a home for polytopes of this description in the Complex Euclidean space, where A and B represent mirrors in the Wythoff-style construction. The meaning of q is a little harder to grasp from the text.

Coxeter gives the order of $p(q)r$ as $g = \frac{p^2r^2q}{\{(p+r)q-pr(q-2)\}^2}$. The order can also be given more naturally in terms of $Pabc$, where P is a poincaré group, and $a,\ b,\ c$ are the various mirrors placed in this group.

In the case of $p(2q)2$, one can apply the normal laws of symmetry, to resulve $p(q)p$, where q can be even or odd. It is derived that when q is odd, then the mirrors A and B are conjugate. That is, a point exists in the symmetry where A and B are swapped.

Much of what is written on this subject, is a matter of deriving the groups from

12.1 Metric Properties

Coxeter's Regular Complex Polytopes gives the margin-angle for all of the regular polytopes, including the stars, and from this, it is possible to derive a formula for the shortchord of these polygons. This REXX subroutine returns the shortchord square, was reversed hacked from Coxeter's tables to give the correct dihedral angles. The format is complex(3,5,3) for 3{5}3.

```
complex:; procedure; parse arg cp1,cp2,cp3; cpx = cp1*cp3
   if cpx = cp1+cp3 then return a2(cp2)
   cpi = (cp1+cp3)   ; cpi = 2*cpx/(cpx-cpi) ; cpi = a2(cpi)
   cpz = abs(cp1-cp3); cpz = 2*cpx/(cpx-cpz) ; cpz = a2(cpz)
   return (a2(cp2)-cpz)/(cpi-cpz)*4
```

The square of diameter is given by the usual form, that is, $d^2 = 4/(4 - a^2)$

The relation that $d^2(p) \cdot d^2(p\{q\}p$ gives the diameter of the euclidean polytope this equates to, that is $\{3, 3, 5\}$, $\{3, 4, 3\}$ or $\{3, 3, 5\}$. Such is to be expected if these polygons have edges that are polygon-shaped in the larger form.

---

[15] A segmentotope is a polytope where the vertices fall into two layers, separated by a height enabling unit edges. The pre-existing notation was to use description || description. The example of note is cube || icosahedron, which amounts to the same height as `vo5oo3ox&#x`, that is, the cube of unit edge is in an icosahedron of edge $1/\phi$.

[16] 1856, p440

[17] A Cayley diagram is a diagram which shows the members of a group in relation to the definitions. The cante-truncates of polyhedra (ie tCO and tID) are the Cayley diagrams of the octahedral and icosahedral groups.

When the mirrors are from different columns, the class of the resulting figure doubles.

12.2 Mirror-Groups

This is derived from the table of subgroups, and the notion that additional mirrors in a group divide the cell into smaller units. Since each mirror divides the cell-space by a specific number, it is possible to work out the space of symmetry 'without mirrors'. These correspond to the various poincaré tilings in S3.

The Poincaré group represented by the octagonny o3x4x3o, has no regular elements.

Poincaré	Order	5	2	3	4/2	4
Tesseract	8	-	-	c, c	x	
24choron	24	-	y	c	x	z
Twelftych	120	f_5	-	f_3	$f2$	
Shape		Icosa	CO	Dodeca	Oct	Oct

Table 16: Complex Groups

These map by clifford-rotations onto the various polyhedral groups, the exact mirrors form different polyhedra in these groups. The subgroups then correspond to whether the various shapes combined are a subset of the larger ones.

So for example, we note that the 120 group has the 8, but not the 24 group as a subset. There is a subgroup if the the 8 groups are represented in the 120 group, so f_5f_3 contains cc, but not ccx, because the larger group has no presence in that column.

The group of 24 contains the group of 8, the groups 8cc and 8x are the same as 24c and 24x.

The listed groups is slightly larger than Coxeter's list in 'regular polytopes', because it is supposed that some of these groups are degenerate compounds by themselves, in much the same way that `r` is the rectanguloid group, is a single entity in `hr` cubic group.

12.3 A description of the symmetry groups

12.3.1 From Complex Numbers to the Swirlybob

The complex euclidean space is taken to represent euclidean space with complex numbers. A straight line still represents the equation $y = ax + b$, but all of these numbers are complex. We still have ordinary parallelism, in that b can be replaced by any b_i, and likewise one can pass a single straight line through any two points. CE2 represents an important subgroup of E4, in much the same way that the argand diagram CE1 might be represented by E2.

At the origin, we suppose $b = 0$, and thus $y = ax$. To this we introduce a further factor $w = \omega t$, to get $wy = awx$. Every point circles the origin at the same angular speed, without changing the gradients of any line. Note there is no facility to produce reversal, ie to get some $wy = -awx$, and we can thus suppose that given an arrow by ωdt, all points move in an individual and unique direction.

The gradient mapped onto a plane, produces an argand diagram over a. This can be mapped onto a *latitude sphere* by placing a sphere with a diameter from (0,0,0) to (0,0,1), where the first two coordinates are the Real and Imaginary axis, and the third is some height.

A ray is then drawn from a point a to (0,0,1), which strikes the sphere at A. The ray from (0,0,1) to the point on the latitude sphere directly opposite A strikes the plane at a point $(-1/a)$. This is exactly the perpendicular line in ordinary euclidean geometry, and likewise in CE2.

Distances on the sphere represent great arrows that are at an angle half of that distance. So points opposite represent planes at 90 deg and so forth.

The longitude circle completes the picture. The sphere rotating under an isoclinal rotation is following a swirlybob, the longitude is the passing of the day, and the latitude dictates the climate. On a planet, the sun might follow a year-arrow not part of this set of swirls. As such there is just one arrow which it is equidistant to, and in the same direction, and the opposite point be equidistant but in an opposite direction. This creates a south pole, a north pole, a line of tropics and a line of artics, much as one hemisphere of our world. The climate is governed by angles from the poles, and the angles of 'longitude' on the latitude sphere is now a 'season-zone' to replace our 6-month difference in hemispheres.

12.3.2 From Swirlybobs to Poincaré space

The poincaré space is created by imposing a rotational symmetry on the latitude-sphere. When an axis is set up, this creates a rotation around a point on the latitude sphere, which leaves two arrows in place, and the rest of the sphere is rotated by the stated angle.

The operation of various symmetry groups on the latitude sphere creates a repetition of cells on the surface of the glome, which represents the sum images of individual points. For example, the icosahedral symmetry on the latitude sphere, causes the surface to repeat in 120 cells. These cells do not have precise boundaries, but usually the most compact shape is selected.

A *poincaré star* is the image of a point under these operations. The poincaré cell has one point from each star. This is a modulus-like function, like the days of the week. Each week has seven days, but the week can start at any point, eg Sunday or Monday. The effect of a poincaré group is to carry any constellation of points in a cell to their images in each cell.

The poincaré groups are subgroups of the regular symmetries of four dimensions, with the exception of the simplex {3,3,3}. Every polytope constructed in these symmetries will divide exactly into separate stars under the operation of the poincaré group.

So for example, the icosahedral group gives a group of 120, the vertices of the poincaré-star is the {3,3,5}. Any of the 15 polychora described in this group are a compound of {3,3,5}, one per vertex. The rectified {3,3,5} has 720 vertices, and is so six such {3,3,5}s.

The poincaré stars represent the units of various integer systems in quaterions, when the point is carried from the identity. Since it suffices to have only a left and a right quaterion to provide every wheel-rotation, we conclude that two swirlybobs suffice to define rotations in four dimension.

12.3.3 Swirls and Swirlybobs in General

Swirlybobs and poincaré groups exist in every even dimension, and are the underlying basis of the rotations. A swirlybob has an arrow in reference to the rotation of coordinates around a centre, which leaves all line-slopes unchanged. The equations defining a line give $n-1$ equal signs, for exmaple, $z = ay + p = bx + q$, at the origin, $z = ay = bx$. Multiplying through by $w = \omega t$ gives a rotation that orbits the centre in non-crossing directed circles (arrows), and such is called a *swirl*.

One should note that swirls thus defined are not changing direction. Indeed, these are *Clifford parallels*. The swirling comes from when one views objects under motion on a clifford rotation or swirl. Such appear to roll or rotate perpendicular to the direction of motion, relative to a fixed observer. All points are indeed heading in straight lines. Note however that in four and more dimensions, two objects following each other on a great arrow, can by the existence of multiple swirls, each be proceeding in a straight line, yet roll relative to each other.

For a given singular swirl, one might suppose that points are engraved to see collinearity if the several points exist on the same arrow. Such a point is called a *swirlybob*. It represents an equipartition of energy around orthogonal modes, and so is the mode a planet might rotate in. At the surface of the planet, the east-west axis runs along the great arrow through the observer, and the rising sphere (for being N-2 dimensions), is a swirl in that dimension. These swirls at the perpendicular to the great-arrow through the observer, divide the rising half of the sky from the setting half. The stars on this sphere never rise nor set, but their full track is seen on the horizon. The points higher up in the sky rise at an angle θ from the middle sphere, and cumulate at θ at the middle of their passage. the track across the sky is half a circle, the setting directly opposite the rising.

The latitude-sphere is formed by a sphere, running from the zenith to the observer, and in the middle of the sky (ie between where stars rise and set). The stars are mapped according to where the line at cumulation [highest point] to the observer crosses the described sphere. This sphere is the same for all observers, the difference being that the zenith is set to match the latitude of the observer.

The full rotation of the planet is then the product of latitude and longitude. Longitude is in the alignment of the arrows, and latitude is represented as a point on the latitude-sphere. In three dimensions, the latitude-sphere would correspond to the gimbal that runs from pole to pole, whilst the longitude is set by rotating the sphere.

In two dimensions, there are two swirlybobs, clockwise and anticlockwise. The poincaré groups are the rotation groups of polygons. The poincaré group is transferred around the sphere by rotation. The cell represents unique points, which are replicated to the vertices of a *poincaré star*. So in an arc, three points might be carried to give the vertices of three pentagons, where the pentagon is the star.

12.3.4 Poincaré groups

The poincaré groups are created by admitting a rotation group on the latitude sphere. This creates on each great arrow, this creates, for each object around which rotation in the latitude sphere happens, a rotation around the space that the latitude-axis has created. The opposite points of the latitude sphere represent arrows that are completely orthogonal, but in six or more dimensions, this is a single arrow in the orthogonal space, not the complete space.

The effect of these additional rotations, is to carry points to represent a vertex of a solid of that space. This solid has identical spaces, the poincaré groups carries an image to lie in each of these faces without overlap. A point is imaged into the vertices of a poincaré star, any constellation of points will give rise to an equal number of these stars.

Complex euclidean space creates these groups, along with additional *reflections*, which are implemented as wheel-rotations in a space orthogonal to a latitude rotation group.

12.4 From Poincaré space to Complex Polygons

Mirrors in complex polytopes are implemented as wheel rotations, in the same direction as the swirl at vertices of a figure in the latitude-sphere. So the effect of the mirror is among other things, rotate the latitude-sphere around one of its vertices, and thus rotate all-space around the hedrix representing the point on the latitude-sphere.

When applied to real polytopes, the wheel rotation would reverse the parity vector (perpendicular to the object in rotation), and on a half-circle rotation, will point in the opposite direction, reversing the parity of the object. This is exactly what happens under ordinary rotation.

A mirror-rotation is implemented by holding one point on the latitude-sphere (the axle) constant, and allowing the latitude-sphere to rotate around the axis through the still and the opposite point. The opposite point rotates in the same line. The remainder of the great arrows are individually relocated to different great arrows.

As with real polygons, all of these axies pass through the centre, and are thus implemented as points on the latitude-sphere. But unlike the three-dimensional polyhedra, the mirrors are separate and non-reliant on other mirrors. An icosahedral group with order-2 mirrors does not imply order-3 or order-5 mirrors, for example.

For the regular complex polygons, there are four classes of mirror, being those of the Icosahedron, the Cuboctahedron, the Dodecahedron, and the Icosadodecahedron.

12.4.1 The Icosahedron $\mathtt{f}_5$

The icosahedron applied to the latitude-sphere gives rise to 12 mirrors, each of which carry ten vertices and ten edges. In $\{3, 3, 5\}$, the vertices and edges cross the great circles, but only twelve of these great circles are used.

The flag runs from a vertex of a pentagon to the edge centre, which is on a different great arrow. The net effect is that by walking the flags of a polygon, one walks the edge-map of an icosahedron. The number of flags is 600, being five at each pentagon, and five at each vertex. The simple edge of 1, of an icosahedron, gives The $5\{3\}5$, while the longer edges gives rise to the star $5\{5/2\}5$ of density 11.

From the discussion above, we note that the flags are half-edge in length, and the circum-diameter of these polygons are the same as the polyhedron walked, and that the diameter of the polygon representing the edge, times the polyhedron walked in the latitude sphere, is the same as the poincaré polytope.

12.4.2 The Cuboctahedron $\mathtt{y}$

The cuboctahedron produces an order-2 group $\mathtt{y}$. The flags of this group would follow a girthing hexagon, with no means to escape. It needs three mirrors together, such as the vertices of a triangle face, to allow the full polyhedron on the latitude-sphere to be walked.

An alternate polygon that can be walked on the cuboctahedron is the square formed by the diagonals of the cuboctahedron's squares. Both of these will be used when we mix pairs of mirrors.

12.4.3 The Dodecahedron `f3 cc c`

The order-3 mirrors are represented by the dodecahedron, the cube, and the tetrahedron on the latitude-sphere. Because these contain all of the smaller figures, these mirror-sets are also subgroups of each other.

These in turn give rise to polygons of $3\{P\}3$, for $P = 5,\ 4,\ 3,\ 5/2$ of densities 1, 5, 15 and 19 resp. The stars of density 5 and 15 are compounds formed from lesser polygons. In the $3\{5\}3$, the great arrows runs through the apex and opposite triangle of the tetrahedra of the $\{3, 3, 5\}$.

The $3\{4\}3$ has the vertices of a $\{3, 4, 3\}$ and the mirrors of `cc` form a pair of tetrahedra inside a cube. The two mirrors are different sets, because the middle number '4' does not allow transport of like mirrors across the branch. The mirrors are opposite each other, and one set of mirrors contain six girthing hexagons, while the other passes through the six triangles that separate the octahedral faces not involving the girthing triangle.

The $3\{3\}3$ is the mirror group `c`. Its vertices belong to that of the $\{3, 3, 4\}$, each mirror passes through two vertices and two edges of that polychoron. It is clear that we can not inscribe $3\{3\}3$ in $3\{4\}3$, because the larger figure contains mirrors containing vertices or edges. However, the stellation of $3\{4\}3$ produces $3\{6/2\}3$ which does contain the required mirrors.

12.4.4 The IcosaDodecahedron `f2 z x`

Like the cuboctahedron, the bulk of this group contains order-2 mirrors, which is not enough to walk the surface of the polytopes in question. We require three order-2 mirrors to walk the icosadodecahedron of `f2` or the octahedron of `x`. Because the icosadodecahedron contains the octahedron, the subgroup relation applies. As with the cuboctahedron, we can walk any of the inscribed polygons, thus 10, 6, 4, for the icosahedron, and 4 for the octahedron.

The octahedron also admits a four-fold rotation on its vertices, which allows for order-4 mirrors. Such gives rise to the polyhedron $4\{3\}4$, with 96 flags, falling to the 24 squares of the girthing squares of $\{3, 4, 3\}$

12.5 General Comments

The shortchord of the polygons $p\{q\}p$ is that required to produce a polygon whose diameter exactly matches $\{q, p\}$, the product of the diameters of $\{p\}$ and $\{q, p\}$ exactly match that of the poincare polychoron. When $p = 2$, this yields the square, hexagon and decagon respectively.

The order of these groups are $p \cdot G$, where G is the order of the poincare group order. Thus they have G vertices and edges.

12.5.1 Polygons with two mirrors

Coxeter's definition of regular is of polytopes whose symmetry is transitive on the flags. The vertex and edge points can lie on different mirrors, and such polygons are formed by combining pairs of mirrors of the same poincaré group.

The latitude-polyhedra are expanded to the same diameter, the edges now pass from one polyhedron to another. The group order is of the form for $P\{Q\}R$, P times R times the poincaré group. So $4\{6\}2$ is 8 times the order of the octahedral poincaré group of order 24, that is of order 192.

The effect of expanding the polyhedra to meet the same circum-diameter, gives numbers which do not occur in the octahedral and icosahedral systems. Such are of the nature of directional incommeasurables of the nature of a *multiplication half.* Such groups that arise from these, can not therefore be subgroups of the full reflective group of the poincaré polychoron. Indeed, we shall see the appearance of polygons with 8, 12, 20 and 30 sides, which do not occur in the poincaré polychoron.

When '2' is one of the mirrors, the effect to to produce a polygon whose short-chord is $\sqrt{2 + \sqrt{a}}$, as with real polygons. This leaves only $3\{4\}4$ and $3\{4\}5$

12.5.2 Tegums, Prisms, Wraps, and other rectanguloids

The bulk of the regular complex polytopes, like the real ones, consist of products over the line-segments, leading in unspectacular series of tegums and prisms. The prisms are by repetition of the base, consist of the polygon-as-edges of the prism-product of n bases. For two dimensions, this represents the polygons of the bi-polygon prism. And so forth to higher dimensions. The surtope consist being the appropriate power of (1,p) with the right-hand rooted at points.

The tegum-product is over the vertices of the crossing bases, consist entirely of the drawn product of points, that is, ordinary simplexes of n vertices. In this, the surtope consist is naturally the power over (p,1), with the units at the nulloid.

The {4/2} and the stella octangula, are represented in every dimension by a crossing of p p-gons whose vertices run diagonally around the bi-polygon prism.

The wrap is a usually hyperbolic function, which means that if '2' works in a position, any higher number can be used. A wrap is then an ordinary polygon {p} so placed that it runs a diagonal of the prism.

13 Circle Drawing and Curvature

The common approach to hyperbolic geometry is to treat it as a variety of spherical geometry. The simplest action here is to use inscribed polytopes, along with a crooked ruler, to measure crooked things. A pentagon, done in spherical trigonometry, has its chords in little relation to the sides. As an inscribed polytope, the chords are always $\phi = \frac{1}{2} + \frac{1}{2}\sqrt{5}$ times the edge. This happens to any size.

We might suppose on a globe, the gimbal or supporting arc, is rendered not in degrees but in inches of chord. In place of running from 90° to $-90°$, the chord runs from 0 to 7920 miles, scale measure, such that 180° is twice 60°.

The crooked ruler we use in the hyperbolic space is the horocycle, or line of no curvature. In essence, it's an euclidean line, but because the space is less curved than it, it goes in an arc, and is by no means the shortest route.

13.1 The Art of Drawing Circles

The intersection of a euclidean space and a non-euclidean space is a circle, but points on this circle have both geometries. A circle surrounding a pentagon, will place the five vertices at 72°, regardless of that circle's relative size. Note however, that the centre of the circle is not on the circumference, and thus we need to walk the various circles, so that the centre might be on a circle.

The chord connecting two points is invariant to what circles it falls on, and this allows us to transfer measure from circle to circle.

For example, if we suppose the polytope is {4,12}, the first circle drawn would be around the square. For a radius equal to the edge, the edge of the dodecagon would be $\sqrt{2}$. The next circle drawn would enclose the dodecagon, whose diameter is $\sqrt{6} + \sqrt{2}$. The overall diameter is them $2\sqrt{3} + 2$. But as we saw before, the arithmetic is easier with the squares of lengths, because at least the euclidean right-angle is simply a sum.

Now, the shortchord of the girthing formed on the above figure, of unit edge, is $2\sqrt{3} + 2$. Although this is larger than two, it's because we're using a crooked line, and in-line lengths do not add. Instead, as in the spherical case, it can be used to find the radius of the enclosing sphere, or more usefully, the edge length when flat.

What we find here, does not only to the flat case, but all versions of the polytope {4,12}. Just as a sphere reduces an equatorial pentagon as it moves away from the equator, the bollosphere increases the sizes. Hyperbolic space expands exponentially, and as such, can not be exactly reproduced in any euclidean space. Instead, we are triangulating distances in euclidean measure, even if some of the figures make no sense.

13.2 The Nature of Curvature

The polytopes measured with euclidean geometry, are not size-dependent. Instead, the edge is minimal (hyperbolic) or maximal (spheric), as the figure is set as a tiling in a plane. For those used to euclidean geometry, it might appear strange that doubling the edge length might make the figure have right-angle corners. But this is the nature of circle-drawing in all non-euclidean geometries. The equatorial pentagon has angles of 180°, but at various latitudes, this can be 120° or at a zero-edge limit, 108°. Likewise, it is possible for the angle to be less than 108° in hyperbolic space. The tiling of {5,4} has right-angle pentagons.

When a cloth is placed on a surface, it will either ripple or ruffle if the curvature is greater or smaller than the surface. For example, a skull-cap sits flat on someone's head, but on the table, it would ripple. The circumferences of circles is less than 2π radii, and so it will produce ripples radiating from the centre, as much as water ripples away from a dropped stone. When the same cloth is placed on a smaller sphere, the circumference is longer than that of the surface, and so it will tend to bunch up radially, or ruffle.

We might note that in non-euclidean geometry *flat* and *zero-curvature* have different meanings. *Flat* means that the subspace has the same curvature as all-space. A geodesic on a sphere is flat, even though it is a circle. It is concentric with the sphere. *Zero-curvature* means that the subspace has a euclidean geometry, is is of the same curvature as our crooked ruler.

The spaces we are dealing with are homogenous and isotopic. This means that there is no grain or implied direction, and that rotation and motion are not either direction nor position dependent.

13.3 Gravity - an aside

The various theories of relativity talk of *space-time*. The geometry of this is that of Minkowski, which features four axies, one of which is treated as being $ict = ct\sqrt{-1}$. One might note that the numbers are all real, and complex values like $2 + i$ are not encountered.

Space-time exists in ordinary relativity: the plot of cars travelling along a road and when they might meet, is an example of space-time. Space on one axis, time on the other. A stack of slides for an animation represents two space-axies and one time axis. The characters snake from bottom to top of the slides, each motion being imperceptible from the next. Turning to slide 108 allows us to see the state at that point, but the objects, even for being cartoons, are not representation of the character we might deal with, but a twisting prism with sections in motion. We are not dealing with this.

Another model is that of a table, with various depressions in them, and heavy spheres sitting in these. This is not a space-time graph. Instead, it represents two dimensions of space, coupled with a vertical potential energy, which gravity and height do a very good job at. What happens is the table is meant to show smaller spheres cued as in billiards, their motion being affected by the curvature in the table.

When it is said space is curved, what is meant is that a circle in degrees, has a varying number of inches per degree. Space being in tension, then has more inches per degree towards a heavy mass than away from it. It is possible to take this model, and derive a space of the right curvature.

Given $E = mc^2$ and $F = \frac{GMm}{r^2}$ as the large-scale approximates, one might imagine that gravity is the excess of length inside to outside of the sphere. It is not all that hard to show that any circle drawn around M has a circumference increased by $2\pi GM/c^2$. In turn this is distributed over R, which gives a pulling-in potential of GM/c^2R, and the gradient as GM/c^2R^2.

You can make such a space, by imaging that for a circle of diameter r, the centre has been pulled in to make a circumference of $4\pi r$. Circles outside the mass would have a circumference of $2\pi R$ but $2\pi R + r$. Space would ruffle in the euclidean. Putting values of r in, gives for the earth, a value of 4.432,96 mm. Such would be imperceptible over 6,000 km.

Space, as far as we can tell, is essentially zero-curvature. Given various models of it suggest that curvature might vary from positive to negative over small regions, the present model is more apt than the three Euclid-style versions (with various replacements for postulate V).

One might also note that geometry by itself can not produce a force. It supposes some distribution of point-like forces to create the force.

14 Hyperbolic Geometry